GOVERNMENT OF TAMIL NADU

HIGHER SECONDARY FIRST YEAR

MATHEMATICS

VOLUME - II

A publication under Free Textbook Programme of Government of Tamil Nadu

Department of School Education

Untouchability is Inhuman and a Crime

Government of Tamil Nadu

First Edition - 2018

Revised Edition - 2019, 2020

(Published under New Syllabus)

Content Creation

State Council of Educational Research and Training

Printing & Publishing

Tamil NaduTextbook and Educational Services Corporation

www.textbooksonline.tn.nic.in

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Chapter 7 Matrices and Determinants

"Mathematics is the music of reason"

- Sylvester

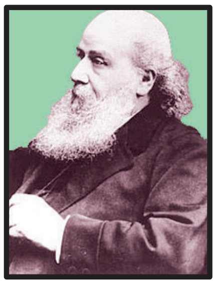
7.1 Introduction

The beginnings of matrices and determinants go back to the second century BC although traces can be seen back to the fourth century BC. However, it was not until near the end of the seventeenth century that the ideas reappeared and development really got underway. It is not surprising that the beginnings of matrices and determinants should arise through the study of systems of linear equations. The Babylonians studied problems which lead to simultaneous linear equations and some of these are preserved in clay tablets which survive till now.

The evolution of the theory of **'matrices'** is the result of attempts to obtain compact and simple methods for solving systems of linear equations. It also began with the study of transformations of geometric objects. In 1850, it was **James Joseph Sylvester** an English Mathematician and lawyer, coined the word '**Matrix**' (originally from Latin: M$\bar{a}$ter means Mother – Collin's Dictionary). Matrices are now one of the most powerful tools in mathematics.

Generally, a matrix is nothing but a rectangular array of objects. These matrices can be visualised in day-to-day applications where we use matrices to represent a military parade or a school assembly or vegetation.

The term **'determinant'** was first coined by **Carl F Gauss** in **Disquisitiones arithmeticae** (1801) while studying quadratic forms. But the concept is not the same as that of modern day determinant. In the same work Gauss laid out the coefficients of his quadratic forms in rectangular arrays where he described matrix multiplication.



Sylvester (1814 - 1897)

It was Cauchy (in 1812) who used *determinant* in its modern sense and studied it in detail. He reproved the earlier results and gave new results of his own on minors and adjoints. It was **Arthur Cayley** whose major contribution was in developing the algebra of matrices and also published the theory of determinants in 1841. In that paper he used two vertical lines on either side of the array to denote the determinant, a notation which has now become standard. In 1858, he published

Memoir on the theory of matrices which was remarkable for containing the first abstract definition of a matrix. He showed that the coefficient arrays studied earlier for quadratic forms and for linear transformations were special cases of his general concept. They simplify our work to a great extent when compared with other straight forward methods which would involve tedious computation. The mathematicians **James Joseph Sylvester** (1814 – 1897), **William Rowan Hamilton** (1805 – 1865), and **Arthur Cayley** (1821 – 1895) played important roles in the development of matrix theory. English mathematician **Cullis** was the first to use modern bracket notation for matrices in 1913. The knowledge of matrices is absolutely necessary not only within the branches of mathematics but also in other areas of science, genetics, economics, sociology, modern psychology and industrial management.

Matrices are also useful for representing coefficients in systems of linear equations. Matrix notations and operations are used in electronic spreadsheet programs on computers, which in turn are used in different areas of business like budgeting, sales projection, cost estimation, and in science, for analyzing the results of an experiment etc.

Interestingly, many geometric operations such as magnification, rotation and reflection through a plane can also be represented mathematically by matrices. Economists use matrices for social accounting, input-output tables and in the study of inter-industry economics. Matrices are also used in communication theory and network analysis in electrical engineering. They are also used in Cryptography.

In this chapter, we now first discuss matrices and their various properties. Then we continue to study determinants, basic properties, minors and their cofactors. Here we now restrict the discussion up to determinants of order 3 only.

Learning Objectives

On completion of this chapter, the students are expected to

- visualise difficult problems in a simple manner in terms of matrices.
- understand different types of matrices and algebra of matrices.
- compute determinant values through expansion and using properties of determinants.
- apply the concepts of matrices and determinants to find the area of a triangle and collinearity of three points.

7.2 Matrices

A **matrix** is a rectangular array or arrangement of entries or elements displayed in rows and columns put within a square bracket [ ].

In general, the entries of a matrix may be real or complex numbers or functions of one variable (such as polynomials, trigonometric functions or a combination of them) or more variables or any other object. Usually, matrices are denoted by capital letters *A, B, C, ...* etc. In this chapter the entries of matrices are restricted to either real numbers or real valued functions on real variables.

General form of a matrix

If a matrix A has m rows and n columns, then it is written as

$$A = [a_{ij}]_{m\times n}\ , 1 \le i \le m, 1 \le j \le n.$$

That is,

$$
A=[a_{ij}]_{m\times n}=
\begin{array}{c}
\overrightarrow{\qquad\qquad n \text{ columns}\qquad\qquad} \\
\begin{array}{cccccc}
\text{Column 1} & \text{Column 2} & & \text{Column } j & & \text{Column } n \\
\downarrow & \downarrow & & \downarrow & & \downarrow
\end{array} \\
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\end{array}
\begin{array}{l}
\\ \\
\leftarrow \text{Row 1} \\
\leftarrow \text{Row 2} \\
\\
\leftarrow \text{Row } i \\
\\
\leftarrow \text{Row } m
\end{array}
\; m \text{ rows}
$$

Note that m and n are positive integers.

The following are some examples of matrices :

$$
A=\begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 9 & -8 & 6 \end{bmatrix},\ B=\begin{bmatrix} 7 & -9 & 1.2 & 0 \\ \sin\frac{x}{4} & 2 & x^2 & 4 \\ \cos\frac{x}{2} & 1 & 3 & -6 \end{bmatrix}, \text{and } C=\begin{bmatrix} 1 & 5 & -7 \\ 3.4 & \frac{1}{2} & \sqrt{2} \\ e^2 & -3 & 4 \\ \sqrt{5} & 2 & a \end{bmatrix}.
$$

In a matrix, the horizontal lines of elements are known as **rows** and the vertical lines of elements are known as **columns**. Thus A has 3 rows and 3 columns, B has 3 rows and 4 columns, and C has 4 rows and 3 columns.

Definition 7.1

If a matrix A has m rows and n columns then the **order or size** of the matrix A is defined to be $m \times n$ (read as m by n).

The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called **elements or entries** of the matrix $A = [a_{ij}]_{m\times n}$. The element a_{ij} is common to i^{th} row and j^{th} column and is called $(i, j)^{\text{th}}$ element of A. Observe that the i^{th} row and j^{th} column of A are $1\times n$ and $m\times 1$ matrices respectively and are given by $[a_{i1} \ \ a_{i2} \ \ \ldots \ \ a_{in}]$ and $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

We shall now visualize the representation and construction of matrices for simplifying day-to-day problems.

Illustration 7.1

Consider the marks scored by a student in different subjects and in different terminal examinations. They are exhibited in a tabular form as given below :

	Tamil	English	Mathematics	Science	Social Science
Exam 1	48	71	80	62	55
Exam 2	70	68	91	73	60
Exam 3	77	84	95	82	62

This tabulation represents the above information in the form of matrix. What does the entry in the third row and second column represent?

The above information may be represented in the form of a 3 × 5 matrix A as

$$A=\begin{bmatrix}48 & 71 & 80 & 62 & 55\\ 70 & 68 & 91 & 73 & 60\\ 77 & 84 & 95 & 82 & 62\end{bmatrix}.$$

The entry 84 common to the third row and the second column in the matrix represents the mark scored by the student in English Exam 3.

Example 7.1

Suppose that a matrix has 12 elements. What are the possible orders it can have? What if it has 7 elements?

Solution

The number of elements is the product of number of rows and number of columns. Therefore, we will find all ordered pairs of natural numbers whose product is 12. Thus, all the possible orders of the matrix are $1\times12,\ 12\times1,\ 2\times6,\ 6\times2,\ 3\times4$ and 4×3.

Since 7 is prime, the only possible orders of the matrix are 1 × 7 and 7 × 1.

Example 7.2

Construct a 2 × 3 matrix whose $(i, j)^{\text{th}}$ element is given by

$$a_{ij}=\frac{\sqrt{3}}{2}|2i-3j| \quad (1\le i\le 2,\ 1\le j\le 3).$$

Solution

In general, a 2 × 3 matrix is given by $A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\end{bmatrix}$

By definition of a_{ij}, we easily have $a_{11}=\frac{\sqrt{3}}{2}|2-3|=\frac{\sqrt{3}}{2}$ and other entries of the matrix A may be computed similarly. Thus, the required matrix A is $\begin{bmatrix}\frac{\sqrt{3}}{2} & 2\sqrt{3} & \frac{7\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \sqrt{3} & \frac{5\sqrt{3}}{2}\end{bmatrix}$.

7.2.1 Types of Matrices

Row, Column, Zero matrices

Definition 7.2

A matrix having only one row is called a **row matrix**.

For instance, $A=[A]_{1\times4}=[1\ \ 0\ \ -1.1\ \ \sqrt{2}]$ is a row matrix. More generally, $A=[a_{ij}]_{1\times n}=[a_{1j}]_{1\times n}$ is a row matrix of order $1\times n$.

Definition 7.3

A matrix having only one column is called a **column matrix**.

For instance, $[A]_{4\times 1} = \begin{bmatrix} x+1 \\ x^2 \\ 3x \\ 4 \end{bmatrix}$ is a column matrix whose entries are real valued functions of real variable x. More generally, $A = [a_{ij}]_{m\times 1} = [a_{i1}]_{m\times 1}$ is a column matrix of order $m\times 1$.

Definition 7.4

A matrix $A = [a_{ij}]_{m\times n}$ is said to be a **zero matrix** or **null matrix** or **void matrix** denoted by O if $a_{ij} = 0$ for all values of $1 \le i \le m$ and $1 \le j \le n$.

For instance, $[0]$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are zero matrices of order 1×1, 3×3 and 2×4 respectively.

A matrix A is said to be a **non-zero matrix** if at least one of the entries of A is non-zero.

Square, Diagonal, Unit, Triangular matrices

Definition 7.5

A matrix in which number of rows is equal to the number of columns, is called a **square matrix**. That is, a matrix of order $n \times n$ is often referred to as a square matrix of order n.

For instance, $A = \begin{bmatrix} a & b & c \\ d & c & f \\ g & h & l \end{bmatrix}$ is a square matrix of order 3.

Definition 7.6

In a square matrix $A = [a_{ij}]_{n\times n}$ of order n, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the **principal diagonal** or simply the **diagonal** or **main diagonal** or **leading diagonal elements.**

Definition 7.7

A square matrix $A = [a_{ij}]_{n\times n}$ is called a **diagonal matrix** if $a_{ij} = 0$ whenever $i \ne j$.

Thus, in a diagonal matrix all the entries except the entries along the main diagonal are zero. For instance,

$$A = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 0.5 \end{bmatrix},\ B = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix},\ C = [6],\ \text{and}\ D = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$$

are diagonal matrices of order 3, 2, 1, and n respectively.

Is a square zero matrix, a diagonal matrix?

Definition 7.8

A diagonal matrix whose entries along the principal diagonal are equal, is called a **Scalar matrix**.

That is, a square matrix $A=[a_{ij}]_{n\times n}$ is said to be a scalar matrix if $a_{ij}=\begin{cases} c & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$

where c is a fixed number. For instance,

$$A=\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}, B=\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}, C=\begin{bmatrix} \sqrt{3} \end{bmatrix} \text{ and } D=\begin{array}{c} \begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix} \end{array}\begin{array}{c} R_1 \\ R_2 \\ \\ R_n \end{array}$$

are scalar matrices of order 3, 2, 1, and n respectively.

Observe that any square zero matrix can be considered as a scalar matrix with scalar 0.

Definition 7.9

A square matrix in which all the diagonal entries are 1 and the rest are all zero is called a **unit matrix**. Thus, a square matrix $A=[a_{ij}]_{n\times n}$ is said to be a unit matrix if $a_{ij}=\begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$.

We represent the unit matrix of order n as I_n. For instance,

$$I_1=[1], I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } I_n=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{array}{c} R_1 \\ R_2 \\ \\ R_n \end{array}$$

are unit matrices of order 1, 2, 3 and n respectively.

Note 7.1

Unit matrix is an example of a scalar matrix.

There are two kinds of triangular matrices namely upper triangular and lower triangular matrices.

Definition 7.10

A square matrix is said to be an **upper triangular matrix** if all the elements below the main diagonal are zero.

Thus, the square matrix $A=[a_{ij}]_{n\times n}$ is said to be an upper triangular matrix if $a_{ij}=0$ for all $i>j$. For instance,

$$\begin{bmatrix} 4 & 3 & 0 \\ 0 & 7 & 8 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -5 & 2 \\ 0 & 1 \end{bmatrix}, \text{ and } \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} \text{ are all upper triangular matrices.}$$

Definition 7.11

A square matrix is said to be a **lower triangular matrix** if all the elements above the main diagonal are zero.

More precisely, a square matrix $A=[a_{ij}]_{n\times n}$ is said to be a lower triangular matrix if $a_{ij}=0$ for all $i<j$. For instance,

$\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & -5 & 7 \end{bmatrix}$, $\begin{bmatrix} -2 & 0 \\ 9 & -3 \end{bmatrix}$, and $\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ are all lower triangular matrices.

Definition 7.12

A square matrix which is either upper triangular or lower triangular is called a **triangular matrix**.

Observe that a square matrix that is both upper and lower triangular simultaneously will turn out to be a diagonal matrix.

7.2.2 Equality of Matrices

Definition 7.13

Two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ are **equal** (written as $A = B$) if and only if

(i) both A and B are of the same order

(ii) the corresponding entries of A and B are equal. That is, $a_{ij}=b_{ij}$ for all i and j.

For instance, if

$$\begin{bmatrix} x & y \\ u & v \end{bmatrix} = \begin{bmatrix} 2.5 & -1 \\ \frac{1}{\sqrt{2}} & \frac{3}{5} \end{bmatrix}, \text{ then we must have } x=2.5, y=-1, u=\frac{1}{\sqrt{2}} \text{ and } v=\frac{3}{5}.$$

Definition 7.14

Two matrices A and B are called **unequal** if either of condition (i) or (ii) of Definition 7.13 does not hold.

For instance, $\begin{bmatrix} 4 & -3 \\ 0 & 8 \end{bmatrix} \neq \begin{bmatrix} 8 & -5 \\ 0 & 4 \end{bmatrix}$ as the corresponding entries are not equal. Also $\begin{bmatrix} 4 & -3 \\ 0 & 8 \end{bmatrix} \neq \begin{bmatrix} 5 & -8 \\ 3 & 4 \\ 6 & 7 \end{bmatrix}$ as the orders are not the same.

Example 7.3

Find x, y, a, and b if $\begin{bmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 4 \\ 5 & -5 & -3 \end{bmatrix}$.

Solution

As the orders of the two matrices are same, they are equal if and only if the corresponding entries are equal. Thus, by comparing the corresponding elements, we get

$3x+4y=2,\ \ x-2y=4,\ \ a+b=5,$ and $2a-b=-5.$

Solving these equations, we get $x=2,\ y=-1,\ a=0,$ and $b=5.$

7.2.3 Algebraic Operations on Matrices

Basic operations on matrices are

(1) multiplication of a matrix by a scalar,

(2) addition/subtraction of two matrices, and

(3) multiplication of two matrices.

There is no concept of dividing a matrix by another matrix and thus, the operation $\frac{A}{B}$, where A and B are matrices, is not defined.

(1) Multiplication of a matrix by a scalar

For a given matrix $A=[a_{ij}]_{m\times n}$ and a scalar k, we define a new matrix $kA=[b_{ij}]_{m\times n}$, where $b_{ij}=ka_{ij}$ for all i and j.

For instance, if $A=\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, then $kA=\begin{bmatrix} ka & kb & kc \\ kd & ke & kf \end{bmatrix}$

In particular if $k=-1$, we obtain $-A=[-a_{ij}]_{m\times n}$. This $-A$ is called **negative** of the matrix A. Don't say $-A$ as a negative matrix.

(2) Addition and Subtraction of two matrices

If A and B are two matrices of the same order, then their sum denoted by $A+B$, is again a matrix of same order, obtained by adding the corresponding entries of A and B.

More precisely, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$ are two matrices, then the sum $A+B$ of A and B is a matrix given by

$A+B=[c_{ij}]_{m\times n}$ where $c_{ij}=a_{ij}+b_{ij}$ for all i and j.

Similarly subtraction $A-B$ is defined as $A-B=A+(-1)B$.

That is, $A-B=[d_{ij}]_{m\times n}$, where $d_{ij}=a_{ij}-b_{ij}$ $\forall$ i and j. (The symbol $\forall$ denotes for every or for all).

Note 7.2

(i) If A and B are not of the same order, then $A+B$ and $A-B$ are not defined.

(ii) The addition and subtraction can be extended to any finite number of matrices.

Example 7.4

Compute $A+B$ and $A-B$ if

$$A=\begin{bmatrix} 4 & \sqrt{5} & 7 \\ -1 & 0 & 0.5 \end{bmatrix} \text{ and } B=\begin{bmatrix} \sqrt{3} & \sqrt{5} & 7.3 \\ 1 & \frac{1}{3} & \frac{1}{4} \end{bmatrix}.$$

Solution

By the definitions of addition and subtraction of matrices, we have

$$A+B=\begin{bmatrix} 4+\sqrt{3} & 2\sqrt{5} & 14.3 \\ 0 & \frac{1}{3} & \frac{3}{4} \end{bmatrix} \text{ and } A-B=\begin{bmatrix} 4-\sqrt{3} & 0 & -0.3 \\ -2 & -\frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

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Example 7.5

Find the sum $A+B+C$ if A, B, C are given by

$$A=\begin{bmatrix}\sin^2\theta & 1\\ \cot^2\theta & 0\end{bmatrix},\ B=\begin{bmatrix}\cos^2\theta & 0\\ -\operatorname{cosec}^2\theta & 1\end{bmatrix} \text{ and } C=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

Solution

By the definition of sum of matrices, we have

$$A+B+C=\begin{bmatrix}\sin^2\theta+\cos^2\theta+0 & 1+0-1\\ \cot^2\theta-\operatorname{cosec}^2\theta-1 & 0+1+0\end{bmatrix}=\begin{bmatrix}1 & 0\\ -2 & 1\end{bmatrix}.$$

Example 7.6

Determine $3B+4C-D$ if B, C, and D are given by

$$B=\begin{bmatrix}2 & 3 & 0\\ 1 & -1 & 5\end{bmatrix},\ C=\begin{bmatrix}-1 & -2 & 3\\ -1 & 0 & 2\end{bmatrix},\ D=\begin{bmatrix}0 & 4 & -1\\ 5 & 6 & -5\end{bmatrix}.$$

Solution

$$3B+4C-D=\begin{bmatrix}6 & 9 & 0\\ 3 & -3 & 15\end{bmatrix}+\begin{bmatrix}-4 & -8 & 12\\ -4 & 0 & 8\end{bmatrix}+\begin{bmatrix}0 & -4 & 1\\ -5 & -6 & 5\end{bmatrix}=\begin{bmatrix}2 & -3 & 13\\ -6 & -9 & 28\end{bmatrix}.$$

Example 7.7

Simplify :

$$\sec\theta\begin{bmatrix}\sec\theta & \tan\theta\\ \tan\theta & \sec\theta\end{bmatrix}-\tan\theta\begin{bmatrix}\tan\theta & \sec\theta\\ \sec\theta & \tan\theta\end{bmatrix}.$$

Solution

If we denote the given expression by A, then using the scalar multiplication rule, we get

$$A=\begin{bmatrix}\sec^2\theta & \sec\theta\tan\theta\\ \sec\theta\tan\theta & \sec^2\theta\end{bmatrix}-\begin{bmatrix}\tan^2\theta & \tan\theta\sec\theta\\ \sec\theta\tan\theta & \tan^2\theta\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

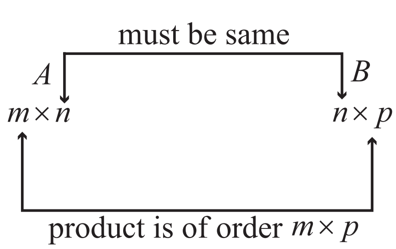
(3) Multiplication of matrices

Definition 7.15

A matrix A is said to be **conformable for multiplication** with a matrix B if the number of columns of A is equal to the number of rows of B.

That is, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are given two matrices, then the product of matrices A and B is denoted by AB and its order is $m\times p$.

The order of AB is $m\times p=$ (number of rows of A) $\times$ (number of columns of B).



If $A = [a_1 \;\; a_2 \;\; \ldots \;\; a_n]_{1\times n}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n\times 1}$,then AB is a matrix of order 1 × 1, that gives a single element which is defined by $AB = [a_1 \;\; a_2 \;\; \ldots \;\; a_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [a_1b_1 + a_2b_2 + \ldots + a_nb_n] = \left[\sum_{k=1}^{n} a_k b_k\right].$

For instance,

$$[1 \;\; 2 \;\; 3]\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} = [1(-2) + 2(3) + 3(5)] = [-2 + 6 + 15] = [19].$$

In general,

if $A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ and $B = [b_{ij}]_{n\times p} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$ then

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix}$$

and the product $AB = [c_{ij}]_{m\times p} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{bmatrix}$,

where $c_{ij} = [a_{i1} \;\; a_{i2} \;\; \ldots \;\; a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = \sum_{k=1}^{n} a_{ik}b_{kj}$, since c_{ij} is an element.

Example 7.8

If $A = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}$, compute A^2.

Solution

$$A^2 = AA = \begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix}\begin{bmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

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where $c_{11} = [0 \;\; c \;\; b]\begin{bmatrix} 0 \\ c \\ b \end{bmatrix} = 0\cdot 0 + c\cdot c + b\cdot b = c^2 + b^2$ and other elements c_{ij} may be computed similarly.

Finally, we easily obtain that

$$A^2 = \begin{bmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{bmatrix} = \begin{bmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{bmatrix}.$$

Example 7.9

Solve for x if $[x \;\; 2 \;\; -1]\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O.$

Solution

$$[x \;\; 2 \;\; -1]\begin{bmatrix} 1 & 1 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

That is, $$[x-2+1 \quad x-8+1 \quad 2x+2+2]\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

$$[x-1 \quad x-7 \quad 2x+4]\begin{bmatrix} x \\ 2 \\ 1 \end{bmatrix} = O$$

$$x(x-1)+2(x-7)+1(2x+4) = 0$$

$$x^2+3x-10 = 0 \;\Rightarrow\; x = -5, 2.$$

Note 7.3

We have the following important observations:

(1) If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$, and $m \neq p$, then the product AB is defined but not BA.

(2) The fundamental properties of real numbers namely,

$ab = ba \quad \forall a, b \in \mathbb{R}$

$ab = ac \Rightarrow b = c \;\forall\; a, b, c \in \mathbb{R},\;\; a \neq 0$

$ab = 0 \Rightarrow a = 0$ or $b = 0 \;\;\forall\;\; a, b \in \mathbb{R}.$

Can we discuss these in matrices also?

(i) Even if AB and BA are defined, then $AB = BA$ is not necessarily true.

For instance, we consider

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

and observe that $AB \neq BA$, since

$$AB = \begin{bmatrix} 5 & 0 \\ 4 & -2 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$$

In this case we say that A and B do not commute (with respect to multiplication)

Observe that $AB = BA$ is also possible. For instance,

$$\text{if } A = \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \text{ then, } AB = BA = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

In this case we say that A and B commute with respect to multiplication.

(ii) Cancellation property does not hold for matrix multiplication. That is, $A \neq O$, B, and C are three square matrices of same order $n \times n$ with $n > 1$, then $AB = AC$ does not imply $B = C$ and $BA = CA$ does not imply $B = C$.

As a simple demonstration of these facts, we observe that for instance,

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but $$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix}.$$

(iii) It is possible that $AB = O$ with $A \neq O$ and $B \neq O$; Equivalently, $AB = O$ is not necessarily imply either $A = O$ or $B = O$. The following relation demonstrates this fact :

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(3) In general, for any two matrices A and B which are conformable for addition and multiplication, for the below operations, we have

- $(A \pm B)^2$ need not be equal to $A^2 \pm 2AB + B^2$
- $A^2 - B^2$ need not be equal to $(A+B)(A-B)$.

Example 7.10

If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 0 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -3 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$ find AB and BA if they exist.

Solution

The order of A is 3×3 and the order of B is 3×2. Therefore the order of AB is 3×2.

A and B are conformable for the product AB. Call $C = AB$. Then,

c_{11} = (first row of A) (first column of B)

$$\Rightarrow \; c_{11} = [1 \;\; -1 \;\; 2]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 1+1+2 = 4, \text{ since } c_{11} \text{ is an element.}$$

Similarly $c_{12} = 0, c_{21} = 0, c_{22} = 13, c_{31} = 7, c_{32} = 5.$

Therefore, $AB = C = [c_{ij}] = \begin{bmatrix} 4 & 0 \\ 0 & 13 \\ 7 & 5 \end{bmatrix}$

The product BA does not exist, because the number of columns in B is not equal to the number of rows in A.

Example 7.11

A fruit shop keeper prepares 3 different varieties of gift packages. Pack-I contains 6 apples, 3 oranges and 3 pomegranates. Pack-II contains 5 apples, 4 oranges and 4 pomegranates and Pack –III contains 6 apples, 6 oranges and 6 pomegranates. The cost of an apple, an orange and a pomegranate respectively are ₹ 30, ₹ 15 and ₹ 45. What is the cost of preparing each package of fruits?

Solution

Cost matrix $A = [30 \;\; 15 \;\; 45]$, Fruit matrix $B = \begin{array}{c} \begin{array}{ccc} \text{P-I} & \text{P-II} & \text{P-III} \end{array} \\ \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} \end{array} \begin{array}{l} \text{Apples} \\ \text{Oranges} \\ \text{Pomegranates} \end{array}$

Cost of packages are obtained by computing AB. That is, by multiplying cost of each item in A (cost matrix A) with number of items in B (Fruit matrix B).

$$AB = [30 \;\; 15 \;\; 45] \begin{bmatrix} 6 & 5 & 6 \\ 3 & 4 & 6 \\ 3 & 4 & 6 \end{bmatrix} = [360 \;\; 390 \;\; 540]$$

Pack-I cost ₹ 360, Pack-II cost ₹ 390, Pack-III costs ₹ 540.

7.2.4 Properties of Matrix Addition, Scalar Multiplication and Product of Matrices

Let A, B, and C be three matrices of same order which are conformable for addition and a, b be two scalars. Then we have the following:

(1) $A + B$ yields a matrix of the same order

(2) $A + B = B + A$ (Matrix addition is commutative)

(3) $(A + B) + C = A + (B + C)$ (Matrix addition is associative)

(4) $A + O = O + A = A$ (O is additive identity)

(5) $A + (-A) = O = (-A) + A$ ($-A$ is the additive inverse of A)

(6) $(a + b)A = aA + bA$ and $a(A + B) = aA + aB$

(7) $a(bA) = (ab)A$, $1A = A$ and $0A = O$.

Properties of matrix multiplication

Using the algebraic properties of matrices we have,

- If A, B, and C are three matrices of orders $m \times n$, $n \times p$ and $p \times q$ respectively, then $A(BC)$ and $(AB)C$ are matrices of same order $m \times q$ and

  $A(BC) = (AB)C$ (Matrix multiplication is associative).

- If A, B, and C are three matrices of orders $m \times n$, $n \times p$, and $n \times p$ respectively, then $A(B + C)$ and $AB + AC$ are matrices of the same order $m \times p$ and

  $A(B + C) = AB + AC$. (Matrix multiplication is left distributive over addition)
- If A, B, and C are three matrices of orders $m \times n$, $m \times n$, and $n \times p$ respectively, then $(A + B)C$ and $AC + BC$ are matrices of the same order $m \times p$ and

  $(A + B)C = AC + BC$. (Matrix multiplication is right distributive over addition).
- If A, B are two matrices of orders $m \times n$ and $n \times p$ respectively and α is scalar, then $\alpha(AB) = A(\alpha B) = (\alpha A)B$ is a matrix of order $m \times p$.
- If I is the unit matrix, then $AI = IA = A$ (I is called multiplicative identity).

7.2.5 Operation of Transpose of a Matrix and its Properties

Definition 7.16

The **transpose** of a matrix is obtained by interchanging rows and columns of A and is denoted by A^T.

More precisely, if $A = [a_{ij}]_{m\times n}$, then $A^T = [b_{ij}]_{n\times m}$, where $b_{ij} = a_{ji}$ so that the $(i, j)^{\text{th}}$ entry of A^T is a_{ji}.

For instance,

$$A = \begin{bmatrix} 1 & \sqrt{2} & 4 \\ -8 & 0 & 0.2 \end{bmatrix} \text{ implies } A^T = \begin{bmatrix} 1 & -8 \\ \sqrt{2} & 0 \\ 4 & 0.2 \end{bmatrix}$$

We state a few basic results on transpose whose proofs are straight forward.

For any two matrices A and B of suitable orders, we have

(i) $(A^T)^T = A$ (ii) $(kA)^T = kA^T$ (where k is any scalar)

(iii) $(A+B)^T = A^T + B^T$ (iv) $(AB)^T = B^T A^T$ (reversal law on transpose)

Example 7.12

If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$,

verify (i) $(AB)^T = B^T A^T$ (ii) $(A+B)^T = A^T + B^T$ (iii) $(A-B)^T = A^T - B^T$ (iv) $(3A)^T = 3A^T$

Solution

(i) $$AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \quad \ldots (1)$$

$$B^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}, \; A^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), $(AB)^T = B^T A^T$.

(ii) $$A + B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \quad \ldots(3)$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \quad \ldots (4)$$

From (3) and (4), $(A+B)^T = A^T + B^T$.

(iii) $$A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$(A-B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad \ldots (5)$$

$$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \quad \ldots (6)$$

From (5) and (6) $(A-B)^T = A^T - B^T$.

(iv) $$3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$$

$$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} = 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} = 3(A^T).$$

7.2.6 Symmetric and Skew-symmetric Matrices

Definition 7.17

A square matrix A is said to be **symmetric** if $A^T = A$.

That is, $A = [a_{ij}]_{n\times n}$ is a symmetric matrix, then $a_{ij} = a_{ji}$ for all i and j.

For instance, $A = \begin{bmatrix} 3 & -6 & 9 \\ -6 & 8 & 5 \\ 9 & 5 & 2 \end{bmatrix}$ is a symmetric matrix since $A^T = A$.

Observe that transpose of A^T is the matrix A itself. That is $\left(A^T\right)^T = A$.

Definition 7.18

A square matrix A is said to be **skew-symmetric** if $A^T = -A$.

If $A = [a_{ij}]_{n\times n}$ is a skew-symmetric matrix, then $a_{ij} = -a_{ji}$ for all i and j.

Now, if we put $i = j$, then $2a_{ii} = 0$ or $a_{ii} = 0$ for all i. This means that all the diagonal elements of a skew-symmetric matrix are zero.

For instance, $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$ is a skew-symmetric matrix since $A^T = -A$.

It is interesting to note that any square matrix can be written as the sum of symmetric and skew-symmetric matrices.

Theorem 7.1

For any square matrix A with real number entries, $A + A^T$ is a symmetric matrix and $A - A^T$ is a skew-symmetric matrix.

Proof

Let $B = A + A^T$.

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B.$$

This implies $A + A^T$ is a symmetric matrix.

Next, we let $C = A - A^T$. Then we see that

$$C^T = (A + (-A^T))^T = A^T + (-A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T) = -C$$

This implies $A - A^T$ is a skew-symmetric matrix.

Theorem 7.2

Any square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

Proof

Let A be a square matrix. Then, we can write

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T).$$

From Theorem 7.1, it follows that $(A + A^T)$ and $(A - A^T)$ are symmetric and skew-symmetric matrices respectively. Since $(kA)^T = kA^T$, it follows that $\frac{1}{2}(A + A^T)$ and $\frac{1}{2}(A - A^T)$ are symmetric and skew-symmetric matrices, respectively. Now, the desired result follows.

Note 7.4

A matrix which is both symmetric and skew-symmetric is a zero matrix.

Example 7.13

Express the matrix $A=\begin{bmatrix}1 & 3 & 5\\ -6 & 8 & 3\\ -4 & 6 & 5\end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrices.

Solution

$$A=\begin{bmatrix}1 & 3 & 5\\ -6 & 8 & 3\\ -4 & 6 & 5\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}1 & -6 & -4\\ 3 & 8 & 6\\ 5 & 3 & 5\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A^T)=\frac{1}{2}\begin{bmatrix}2 & -3 & 1\\ -3 & 16 & 9\\ 1 & 9 & 10\end{bmatrix}$

Now $P^T=\frac{1}{2}\begin{bmatrix}2 & -3 & 1\\ -3 & 16 & 9\\ 1 & 9 & 10\end{bmatrix}=P$

Thus, $P=\frac{1}{2}(A+A^T)$ is a symmetric matrix.

Let $Q=\frac{1}{2}(A-A^T)$

$$=\frac{1}{2}\begin{bmatrix}0 & 9 & 9\\ -9 & 0 & -3\\ -9 & 3 & 0\end{bmatrix}$$

Then $Q^T=\frac{1}{2}\begin{bmatrix}0 & -9 & -9\\ 9 & 0 & 3\\ 9 & -3 & 0\end{bmatrix}=-Q$

Thus $Q=\frac{1}{2}(A-A^T)$ is a skew-symmetric matrix.

$$A=P+Q=\frac{1}{2}\begin{bmatrix}2 & -3 & 1\\ -3 & 16 & 9\\ 1 & 9 & 10\end{bmatrix}+\frac{1}{2}\begin{bmatrix}0 & 9 & 9\\ -9 & 0 & -3\\ -9 & 3 & 0\end{bmatrix}$$

Thus A is expressed as the sum of symmetric and skew-symmetric matrices.

EXERCISE 7.1

(1) Construct an $m\times n$ matrix $A=[a_{ij}]$, where a_{ij} is given by

(i) $a_{ij}=\frac{(i-2j)^2}{2}$ with $m=2, n=3$ (ii) $a_{ij}=\frac{|3i-4j|}{4}$ with $m=3, n=4$

(2) Find the values of $p, q, r,$ and s if

$$\begin{bmatrix} p^2-1 & 0 & -31-q^3 \\ 7 & r+1 & 9 \\ -2 & 8 & s-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 \\ 7 & \frac{3}{2} & 9 \\ -2 & 8 & -\pi \end{bmatrix}.$$

(3) Determine the value of $x+y$ if $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$.

(4) Determine the matrices A and B if they satisfy

$$2A-B+\begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix} = 0 \text{ and } A-2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$$

(5) If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then compute A^4.

(6) Consider the matrix $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

(i) Show that $A_\alpha A_\beta = A_{(\alpha+\beta)}$.

(ii) Find all possible real values of α satisfying the condition $A_\alpha + A_\alpha^T = I$.

(7) If $A = \begin{bmatrix} 4 & 2 \\ -1 & x \end{bmatrix}$ and such that $(A-2I)(A-3I) = O$, find the value of x.

(8) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, show that A^2 is a unit matrix.

(9) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $A^3 - 6A^2 + 7A + kI = O$, find the value of k.

(10) Give your own examples of matrices satisfying the following conditions in each case:

(i) A and B such that $AB \neq BA$.

(ii) A and B such that $AB = O = BA$, $A \neq O$ and $B \neq O$.

(iii) A and B such that $AB = O$ and $BA \neq O$.

(11) Show that $f(x)f(y) = f(x+y)$, where $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(12) If A is a square matrix such that $A^2 = A$, find the value of $7A - (I + A)^3$.

(13) Verify the property $A(B + C) = AB + AC$, when the matrices $A, B,$ and C are given by

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}.$$

(14) Find the matrix A which satisfies the matrix relation $A\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}=\begin{bmatrix}-7 & -8 & -9\\2 & 4 & 6\end{bmatrix}$.

(15) If $A^T=\begin{bmatrix}4 & 5\\-1 & 0\\2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 1\\7 & 5 & -2\end{bmatrix}$, verify the following

(i) $(A+B)^T=A^T+B^T=B^T+A^T$ (ii) $(A-B)^T=A^T-B^T$ (iii) $(B^T)^T=B$.

(16) If A is a 3×4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B?

(17) Express the following matrices as the sum of a symmetric matrix and a skew-symmetric matrix:

(i) $\begin{bmatrix}4 & -2\\3 & -5\end{bmatrix}$ and (ii) $\begin{bmatrix}3 & 3 & -1\\-2 & -2 & 1\\-4 & -5 & 2\end{bmatrix}$.

(18) Find the matrix A such that $\begin{bmatrix}2 & -1\\1 & 0\\-3 & 4\end{bmatrix}A^T=\begin{bmatrix}-1 & -8 & -10\\1 & 2 & -5\\9 & 22 & 15\end{bmatrix}$.

(19) If $A=\begin{bmatrix}1 & 2 & 2\\2 & 1 & -2\\x & 2 & y\end{bmatrix}$ is a matrix such that $AA^T=9I$, find the values of x and y.

(20) (i) For what value of x, the matrix $A=\begin{bmatrix}0 & 1 & -2\\-1 & 0 & x^3\\2 & -3 & 0\end{bmatrix}$ is skew-symmetric.

(ii) If $\begin{bmatrix}0 & p & 3\\2 & q^2 & -1\\r & 1 & 0\end{bmatrix}$ is skew-symmetric, find the values of p, q, and r.

(21) Construct the matrix $A=[a_{ij}]_{3\times3}$, where $a_{ij}=i-j$. State whether A is symmetric or skew-symmetric.

(22) Let A and B be two symmetric matrices. Prove that $AB = BA$ if and only if AB is a symmetric matrix.

(23) If A and B are symmetric matrices of same order, prove that

(i) $AB + BA$ is a symmetric matrix.

(ii) $AB - BA$ is a skew-symmetric matrix.

(24) A shopkeeper in a Nuts and Spices shop makes gift packs of cashew nuts, raisins and almonds.

Pack I contains 100 gm of cashew nuts, 100 gm of raisins and 50 gm of almonds.

Pack-II contains 200 gm of cashew nuts, 100 gm of raisins and 100 gm of almonds.

Pack-III contains 250 gm of cashew nuts, 250 gm of raisins and 150 gm of almonds.

The cost of 50 gm of cashew nuts is ₹ 50, 50 gm of raisins is ₹10, and 50 gm of almonds is ₹ 60. What is the cost of each gift pack?

7.3 Determinants

To every square matrix $A = [a_{ij}]$ of order n, we can associate a number called **determinant** of the matrix A.

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then determinant of A is written as $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Note 7.5

(i) Determinants can be defined only for square matrices.

(ii) For a square matrix A, $|A|$ is read as **determinant of A.**

(iii) Matrix is only a representation whereas determinant is a value of a matrix.

7.3.1 Determinants of Matrices of different order

Determinant of a matrix of order 1

Let $A = [a]$ be the matrix of order 1, then the determinant of A is defined as 'a'.

Determinant of a matrix of order 2

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be a matrix of order 2. Then the determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{21}\ a_{12}.$$

Example 7.14

Evaluate : (i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$.

Solution

(i) $\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} = (2 \times 2) - (-1 \times 4) = 4 + 4 = 8.$

(ii) $\begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = (\cos\theta\cos\theta) - (-\sin\theta\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$

Determinant of a Matrix of order 3

We consider the determinant of a 3×3 matrix with entries as real numbers or real valued functions defined on $\mathbb{R}$ and study its properties and discuss various methods of evaluation of certain determinants.

Definition 7.19

Let $A = [a_{ij}]_{3\times3}$ be a given square matrix of order 3. The **minor** of an arbitrary element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is usually denoted by M_{ij}.

Definition 7.20

The **cofactor** is a signed minor. The cofactor of a_{ij} is usually denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

For instance, consider the 3 × 3 matrix defined by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the minors and cofactors of the elements a_{11}, a_{12}, a_{13} are given as follows :

(i) Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Cofactor of a_{11} is $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii) Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor a_{12} is $A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{31}a_{23})$

(iii) Minor of a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Cofactor of a_{13} is $A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$.

Result 7.1 (Laplace Expansion)

For a given matrix $A = [a_{ij}]_{3\times3}$, the sum of the product of elements of the first row with their corresponding cofactors is the **determinant** of A.

That is, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

This can also be written using minors. That is, $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$.

The determinant can be computed by expanding along any row or column and it is important to note that the value in all cases remains the same. For example,

expansion along R_1 is $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.

along R_2 is $|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$.

along C_1 is $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$.

Example 7.15

Compute all minors, cofactors of A and hence compute $|A|$ if $A = \begin{bmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ -3 & 5 & 2 \end{bmatrix}$. Also check that $|A|$ remains unaltered by expanding along any row or any column.

Solution

Minors :

$$M_{11} = \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = -10-30 = -40$$

$$M_{12} = \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} = 8+18 = 26$$

$$M_{13} = \begin{vmatrix} 4 & -5 \\ -3 & 5 \end{vmatrix} = 20-15 = 5$$

$$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6+10 = 16$$

$$M_{22} = \begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} = 2-6 = -4$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} = 5+9 = 14$$

$$M_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18-10 = 8$$

$$M_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 6+8 = 14$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5-12 = -17$$

Cofactors :

$$A_{11} = (-1)^{1+1}(-40) = -40$$

$$A_{12} = (-1)^{1+2}(+26) = -26$$

$$A_{13} = (-1)^{1+3}(5) = 5$$

$$A_{21} = (-1)^{2+1}(16) = -16$$

$$A_{22} = (-1)^{2+2}(-4) = -4$$

$$A_{23} = (-1)^{2+3}(14) = -14$$

$$A_{31} = (-1)^{3+1}(8) = 8$$

$$A_{32} = (-1)^{3+2}(14) = -14$$

$$A_{33} = (-1)^{3+3}(-17) = -17$$

Expanding along R_1 yields

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}\ .$$

$$|A| = 1(-40)+(3)(-26)+(-2)(5) = -128\ . \qquad \ldots (3)$$

Expanding along C_1 yields

$$|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}\ .$$

$$= 1(-40)+4(-16)+-3(8) = -128 \qquad \ldots (4)$$

From (3) and (4), we have

|A| obtained by expanding along R_1 is equal to expanding along C_1 .

Evaluation of determinant of order 3 by using Sarrus Rule (named after the French Mathematician Pierre Frédéic Sarrus)

Let $A = [a_{ij}]_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

B9F35H

Write the entries of Matrix A as follows :

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Then $|A|$ is computed as follows :

$$|A| = \left[a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\right] - \left[a_{33}a_{21}a_{12} + a_{32}a_{23}a_{11} + a_{31}a_{22}a_{13}\right]$$

Example 7.16

Find $|A|$ if $A = \begin{bmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{bmatrix}$.

Solution

$$\begin{vmatrix} 0 & \sin\alpha & \cos\alpha \\ \sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix} = 0M_{11} - \sin\alpha M_{12} + \cos\alpha M_{13}$$

$$= 0 - \sin\alpha(0 - \cos\alpha\sin\beta) + \cos\alpha(-\sin\alpha\sin\beta - 0) = 0.$$

Example 7.17

Compute $|A|$ using Sarrus rule if $A = \begin{bmatrix} 3 & 4 & 1 \\ 0 & -1 & 2 \\ 5 & -2 & 6 \end{bmatrix}$.

Solution

$$\begin{array}{ccccc} 3 & 4 & 1 & 3 & 4 \\ 0 & -1 & 2 & 0 & -1 \\ 5 & -2 & 6 & 5 & -2 \end{array}$$

$$|A| = \left[3(-1)(6)+4(2)(5)+1(0)(-2)\right] - \left[5(-1)(1)+(-2)(2)3+6(0)(4)\right]$$

$$= [-18+40+0]-[-5-12+0] = 22+17 = 39.$$

Note 7.6

For easier calculations, we expand the determinant along a row or column which contains maximum number of zeros.

Determinant of square matrix of order n, $n \geq 4$

The concept of determinant can be extended to the case of square matrix of order $n, n \geq 4$. Let $A = [a_{ij}]_{n \times n}, n \geq 4$.

If we delete the i^{th} row and j^{th} column from the matrix of $A = [a_{ij}]_{n \times m}$, we obtain a determinant of order $(n-1)$, which is called the **minor of the element** a_{ij}. We denote this minor by M_{ij}. The cofactor of the element a_{ij} is defined as $A_{ij} = (-1)^{i+j} M_{ij}$.

Result 7.2

For a given square matrix $A = [a_{ij}]_{n \times n}$ of order n, the sum of the products of elements of the first row with their corresponding cofactors is the determinant of A. That is,

$|A| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} = \sum_{j=1}^{n} a_{1j}A_{1j}$ which, by the definition of cofactors and minors, is same as

$$|A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j},$$

where A_{1j} denotes the cofactor of a_{1j} and M_{1j} denotes the minor of a_{1j}, $j = 1, 2, \ldots, n$.

Note 7.7

(i) If $A = [a_{ij}]_{n \times n}$ then determinant of A can also be denoted as det(A) or *det A* or Δ.

(ii) It can be computed by using any row or column.

7.3.2 Properties of Determinants

We can use one or more of the following properties of the determinants to simplify the evaluation of determinants.

Property 1

The determinant of a matrix remains unaltered if its rows are changed into columns and columns into rows. That is, $|A| = |A^T|$.

Since the row-wise expansion is same as the column-wise expansion, the result holds good.

Property 2

If any two rows / columns of a determinant are interchanged, then the determinant changes in sign but its absolute value remains unaltered.

Verification

$$\text{Let } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$\text{Let } |A_1| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad (\text{since } R_2 \leftrightarrow R_3)$$

$$= a_1(b_3c_2 - b_2c_3) - b_1(a_3c_2 - a_2c_3) + c_1(a_3b_2 - a_2b_3)$$

$$= -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$
$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$
$$= -|A|$$

Therefore, $|A_1| = -|A|$. Thus the property is verified.

Property 3

If there are n interchanges of rows (columns) of a matrix A then the determinant of the resulting matrix is $(-1)^n|A|$.

Property 4

If two rows (columns) of a matrix are identical, then its determinant is zero.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$, with 2nd and 3rd rows are identical.

Interchanging second and third rows, we get $-|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{vmatrix} = |A|$.

$$\Rightarrow 2|A| = 0 \;\Rightarrow |A| = 0.$$

Property 5

If a row (column) of a matrix A is a scalar multiple of another row (or column) of A, then its determinant is zero.

Note 7.8

(i) If all entries of a row or a column are zero, then the determinant is zero.

(ii) The determinant of a triangular matrix is obtained by the product of the principal diagonal elements.

Property 6

If each element in a row (or column) of a matrix is multiplied by a scalar k, then the determinant is multiplied by the same scalar k.

Verification

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Let $|A_1| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2) = k|A|$$
$$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] = k|A|$$
$$\Rightarrow |A_1| = k|A|.$$

Note 7.9

If A is a square matrix of order n, then

(i) $|AB| = |A||B|$

(ii) If $AB = O$ then either $|A| = 0$ or $|B| = 0$.

(iii) $|A^n| = (|A|)^n$

Property 7

If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant is expressed as sum of two or more determinants.

That is, $$\begin{vmatrix} a_1+m_1 & b_1 & c_1 \\ a_2+m_2 & b_2 & c_2 \\ a_3+m_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix}.$$

Verification

By taking first column expansion it can be verified easily.

$$\text{LHS} = (a_1+m_1)(b_2c_3-b_3c_2)-(a_2+m_2)(b_1c_3-b_3c_1)+(a_3+m_3)(b_1c_2-b_2c_1)$$

$$= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)+m_1(b_2c_3-b_3c_2)-m_2(b_1c_3-b_3c_1)$$
$$+m_3(b_1c_2-b_2c_1)$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_3 & b_3 & c_3 \end{vmatrix} = \text{RHS}.$$

Property 8

If, to each element of any row (column) of a determinant the equi-multiples of the corresponding entries of one or more rows (columns) are added or subtracted, then the value of the determinant remains unchanged.

Verification

Let $$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$|A_1| = \begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \left\{ \begin{array}{l} \text{using} \\ \text{Property 7} \end{array} \right.$$

$$|A_1| = |A| + p\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A_1| = |A| + p(0) + q(0) = |A| \quad \text{(using Property 4)}$$

Therefore $|A_1| = |A|$

This property is independent of any fixed row or column.

Example 7.18

If a, b, c and x are positive real numbers, then show that $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (b^x+b^{-x})^2 & (b^x-b^{-x})^2 & 1 \\ (c^x+c^{-x})^2 & (c^x-c^{-x})^2 & 1 \end{vmatrix}$ is zero.

Solution

Applying $C_1 \to C_1 - C_2$, we get $\begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (b^x-b^{-x})^2 & 1 \\ 4 & (c^x-c^{-x})^2 & 1 \end{vmatrix} = 0$, since C_1 and C_3 are proportional.

Example 7.19

Without expanding the determinants, show that $|B| = 2|A|$.

Where $B = \begin{bmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$

Solution

We have
$$|B| = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} (R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1)$$

$$= 2\begin{vmatrix} a & b & c \\ -b & -c & -a \\ -c & -a & -b \end{vmatrix} (R_1 \to R_1 + R_2 + R_3)$$

$$= 2(-1)^2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 2|A|.$$

Example 7.20

Evaluate $\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 2014 & 2017 & 0 \\ 2020 & 2023 & 1 \\ 2023 & 2026 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 2017-2014 & 0 \\ 2020 & 2023-2020 & 1 \\ 2023 & 2026-2023 & 0 \end{vmatrix} = \begin{vmatrix} 2014 & 3 & 0 \\ 2020 & 3 & 1 \\ 2023 & 3 & 0 \end{vmatrix} = 3\begin{vmatrix} 2014 & 1 & 0 \\ 2020 & 1 & 1 \\ 2023 & 1 & 0 \end{vmatrix}$$

$$= -3(2014-2023) = -3(-9) = 27.$$

Example 7.21

Find the value of x if $\begin{vmatrix} x-1 & x & x-2 \\ 0 & x-2 & x-3 \\ 0 & 0 & x-3 \end{vmatrix} = 0$.

Solution

Since all the entries below the principal diagonal are zero, the value of the determinant is $(x-1)(x-2)(x-3) = 0$ which gives $x = 1, 2, 3$.

Example 7.22

Prove that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

Solution

Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\text{LHS} = \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

$$= (y-x)(z-x)[(z+x)-(y+x)].$$

$$= (y-x)(z-x)(z-y).$$

$$= (x-y)(y-z)(z-x) = \text{RHS}.$$

EXERCISE 7.2

(1) Without expanding the determinant, prove that $\begin{vmatrix} s & a^2 & b^2+c^2 \\ s & b^2 & c^2+a^2 \\ s & c^2 & a^2+b^2 \end{vmatrix} = 0$.

(2) Show that $\begin{vmatrix} b+c & bc & b^2c^2 \\ c+a & ca & c^2a^2 \\ a+b & ab & a^2b^2 \end{vmatrix} = 0$.

(3) Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$.

(4) Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

(5) Prove that $\begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 38 & 36 & 2 \end{vmatrix} = 0.$

(6) Show that $\begin{vmatrix} x+2a & y+2b & z+2c \\ x & y & z \\ a & b & c \end{vmatrix} = 0.$

(7) Write the general form of a 3×3 skew-symmetric matrix and prove that its determinant is 0.

(8) If $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$,

prove that a, b, c are in $G.P.$ or α is a root of $ax^2 + 2bx + c = 0$.

(9) Prove that $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$.

(10) If a, b, c are p^{th}, q^{th} and r^{th} terms of an $A.P$, find the value of $\begin{vmatrix} a & b & c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$.

(11) Show that $\begin{vmatrix} a^2+x^2 & ab & ac \\ ab & b^2+x^2 & bc \\ ac & bc & c^2+x^2 \end{vmatrix}$ is divisible by x^4.

(12) If a, b, c are all positive, and are p^{th}, q^{th} and r^{th} terms of a $G.P.$, show that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0.$

(13) Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ if $x, y, z \neq 1$.

(14) If $A = \begin{bmatrix} \frac{1}{2} & \alpha \\ 0 & \frac{1}{2} \end{bmatrix}$, prove that $\sum_{k=1}^{n} \det(A^k) = \frac{1}{3}\left(1 - \frac{1}{4^n}\right)$.

(15) Without expanding, evaluate the following determinants :

(i) $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ (ii) $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

(16) If A is a square matrix and $|A| = 2$, find the value of $|AA^T|$.

(17) If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$, find the value of $|3AB|$.

(18) If $\lambda = -2$, determine the value of $\begin{vmatrix} 0 & 2\lambda & 1 \\ \lambda^2 & 0 & 3\lambda^2+1 \\ -1 & 6\lambda-1 & 0 \end{vmatrix}$.

(19) Determine the roots of the equation $\begin{vmatrix} 1 & 4 & 20 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$.

(20) Verify that det(AB) = (det A) (det B) for $A = \begin{bmatrix} 4 & 3 & -2 \\ 1 & 0 & 7 \\ 2 & 3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 0 \\ 9 & 7 & 5 \end{bmatrix}$.

(21) Using cofactors of elements of second row, evaluate $|A|$, where $A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

7.3.3 Application of Factor Theorem to Determinants.

Theorem 7.3 (Factor Theorem)

If each element of a matrix A is a polynomial in x and if $|A|$ vanishes for $x = a$, then $(x - a)$ is a factor of $|A|$.

Note 7.10

(i) This theorem is very much useful when we have to obtain the value of the determinant in 'factors' form.

(ii) If we substitute b for a in the determinant $|A|$, any two of its rows or columns become identical, then $|A| = 0$, and hence by factor theorem $(a - b)$ is a factor of $|A|$.

(iii) If r rows (columns) are identical in a determinant of order n $(n \geq r)$, when we put $x = a$, then $(x - a)^{r-1}$ is a factor of $|A|$.

(iv) A square matrix (or its determinant) is said to be in cyclic symmetric form if each row is obtained from the first row by changing the variables cyclically.

(v) If the determinant is in cyclic symmetric form and if m is the difference between the degree of the product of the factors (obtained by substitution) and the degree of the product of the leading diagonal elements and if

(1) m is zero, then the required factor is a constant k

(2) m is 1, then the required factor is $k(a + b + c)$ and

(3) m is 2, then the required factor is $k(a^2 + b^2 + c^2) + l\,(ab + bc + ca)$.

Example 7.23

Using Factor Theorem, prove that $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = (x-1)^2(x+9)$.

Solution

$$\text{Let } |A| = \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix}.$$

$$\text{Putting } x = 1, \text{ we get } |A| = \begin{vmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$$

Since all the three rows are identical, $(x-1)^2$ is a factor of $|A|$

$$\text{Putting } x = -9 \text{ in } |A|, \text{we get} |A| = \begin{vmatrix} -8 & 3 & 5 \\ 2 & -7 & 5 \\ 2 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 5 \\ 0 & -7 & 5 \\ 0 & 3 & -5 \end{vmatrix} = 0$$

Therefore $(x+9)$ is a factor of $|A|$ [since $C_1 \to C_1 + C_2 + C_3$].

The product $(x-1)^2(x+9)$ is a factor of $|A|$. Now the determinant is a cubic polynomial in x.

Therefore the remaining factor must be a constant 'k'.

$$\text{Therefore } \begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = k(x-1)^2(x+9).$$

Equating x^3 term on both sides, we get $k = 1$. Thus $|A| = (x-1)^2(x+9)$.

Example 7.24

Prove that $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)\,(y-z)\,(z-x)\,(xy+yz+zx)$.

Solution

$$\text{Let} \qquad |A| = \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}.$$

$$\text{Putting } x = y \textit{ gives} \qquad |A| = \begin{vmatrix} 1 & y^2 & y^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = 0 \quad (\text{since } R_1 \equiv R_2).$$

Therefore $(x-y)$ is a factor.

The given determinant is in cyclic symmetric form in x, y and z. Therefore $(y - z)$ and $(z - x)$ are also factors.

The degree of the product of the factors $(x-y)(y-z)(z-x)$ is 3 and the degree of the product of the leading diagonal elements $1\times y^2 \times z^3$ is 5.

Therefore the other factor is $k(x^2+y^2+z^2)+\ell(xy+yz+zx)$.

Thus $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = [k(x^2+y^2+z^2)+\ell(xy+yz+zx)]\times(x-y)(y-z)(z-x)$.

Putting $x = 0$, $y = 1$ and $z = 2$, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 4 & 8 \end{vmatrix} = \left[k(0+1+4)+\ell(0+2+0)\right](-1)(1-2)(2-0)$$

$$\Rightarrow \qquad (8-4) = [(5k+2\ell)](-1)(-1)(2)$$

$$4 = 10k+4\ell \Rightarrow 5k+2\ell = 2. \qquad \text{... (1)}$$

Putting $x=0, y=-1$ and $z=1$, We get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = [k(2)+\ell(-1)](1)(-2)(1)$$

$$\Rightarrow \qquad [(2k-\ell)(-2)] = 2$$

$$2k-\ell = -1. \qquad \text{... (2)}$$

Solving (1) and (2), we get $k=0, \ell=1$.

Therefore $\begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx).$

Example 7.25

Prove that $|A| = \begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = 2pqr\ (p+q+r)^3.$

Solution :

Taking $p=0$, we get $|A| = \begin{vmatrix} (q+r)^2 & 0 & 0 \\ q^2 & r^2 & q^2 \\ r^2 & r^2 & q^2 \end{vmatrix} = 0.$

Therefore, $(p - 0)$ is a factor. That is, p is a factor.

Since $|A|$ is in cyclic symmetric form in p, q, r and hence q and r also factors.

Putting $p+q+r=0 \Rightarrow q+r=-p\,;\, r+p=-q\,;$ and $p+q=-r$.

$$|A| = \begin{vmatrix} p^2 & p^2 & p^2 \\ q^2 & q^2 & q^2 \\ r^2 & r^2 & r^2 \end{vmatrix} = 0 \text{ since 3 columns are identical.}$$

Therefore, $(p+q+r)^2$ is a factor of $|A|$.

The degree of the obtained factor $pqr\,(p+q+r)^2$ is 5. The degree of $|A|$ is 6.

Therefore, required factor is $k\,(p+q+r)$.

$$\begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix} = k(p+q+r)\,(p+q+r)^2 \times pqr$$

Taking $p=1,\ q=1,\ c=1$, we get

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{vmatrix} = k(1+1+1)^3\ (1)\ (1)\ (1)\,.$$

$4(16-1)-1(4-1)+1(1-4)=27k$

$60-3-3=27\,k \Rightarrow k=2$.

$|A| = 2pqr\,(p+q+r)^3$.

Example 7.26

In a triangle ABC, if $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A(1+\sin A) & \sin B(1+\sin B) & \sin C(1+\sin C) \end{vmatrix} = 0,$

prove that ΔABC is an isosceles triangle.

Solution :

By putting $\sin A = \sin B$, we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin A & 1+\sin C \\ \sin A(1+\sin A) & \sin A(1+\sin A) & \sin C(1+\sin C) \end{vmatrix} = 0$$

That is, by putting $\sin A = \sin B$ we see that, the given equation is satisfied.

Similarly by putting $\sin B = \sin C$ and $\sin C = \sin A$, the given equation is satisfied.

Thus, we have $A = B$ or $B = C$ or $C = A$.

In all cases atleast two angles are equal. Thus the triangle is isosceles.

EXERCISE 7.3

Solve the following problems by using Factor Theorem :

(1) Show that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x-a)^2(x+2a)$.

(2) Show that $\begin{vmatrix} b+c & a-c & a-b \\ b-c & c+a & b-a \\ c-b & c-a & a+b \end{vmatrix} = 8abc$.

(3) Solve $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0.$

(4) Show that $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix} = (a+b+c)\ (a-b)\ (b-c)\ (c-a)$.

(5) Solve $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$.

(6) Show that $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$.

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7.3.4 Product of Determinants

While multiplying two matrices "row-by-column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant (by Property 1). Therefore we can also adopt the following procedures for multiplication of two determinants.

(i) Row by column multiplication rule
(ii) Row by row multiplication rule
(iii) Column by column multiplication rule
(iv) Column by row multiplication rule

Note 7.11

(i) If A and B are square matrices of the same order n, then $|AB| = |A||B|$ holds.
(ii) In matrices, although $AB \neq BA$ in general, we do have $|AB| = |BA|$ always.

Example 7.27

Verify that $|AB| = |A|\ |B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Solution

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow |AB| = 1. \qquad \text{... (1)}$$

$$|A| = \cos^2\theta + \sin^2\theta = 1.$$

$$|B| = \cos^2\theta + \sin^2\theta = 1.$$

$$|A|\,|B| = 1. \qquad \text{... (2)}$$

From (1) and (2), $|AB| = |A|\,|B|$.

Example 7.28

Show that $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2 = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \times \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}.$$

$$= \begin{vmatrix} 0+c^2+b^2 & 0+0+ab & 0+ac+0 \\ 0+0+ab & c^2+0+a^2 & bc+0+0 \\ 0+ac+0 & bc+0+0 & b^2+a^2+0 \end{vmatrix}.$$

$$= \begin{vmatrix} c^2+b^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & b^2+a^2 \end{vmatrix} = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix} = \text{RHS}.$$

Example 7.29

Show that $\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$.

Solution

$$\text{RHS} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times (-1) \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} \quad [\text{In the } 2^{\text{nd}} \text{ determinant } R_2 \leftrightarrow R_3]$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & -b & -c \\ c & a & b \\ b & c & a \end{vmatrix}.$$

Taking row by column method, we get

$$= \begin{vmatrix} -a^2+bc+cb & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab \end{vmatrix}$$

$$= \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix} = \text{RHS}.$$

Example 7.30

Prove that $\begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}$.

Solution

$$\text{LHS} = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}.$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times (-1)\,(-1) \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x \\ -x & -1 & -x \\ -x & -x & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-x^2-x^2 & x-x-x^2 & x-x^2-x \\ x-x-x^2 & x^2-1-x^2 & x^2-x-x \\ x-x^2-x & x^2-x-x & x^2-x^2-1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-2x^2 & -x^2 & -x^2 \\ -x^2 & -1 & x^2-2x \\ -x^2 & x^2-2x & -1 \end{vmatrix}.$$

$$= \text{R.H.S.}$$

7.3.5 Relation between a Determinant and its Cofactor Determinant

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

Let A_1, B_1, C_1 .... be the cofactors of a_1, b_1, c_1 ... in $|A|$.

Hence, the cofactor determinant is $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$

$|A| = a_1A_1 + b_1B_1 + c_1C_1$

Similarly, $|A| = a_2A_2 + b_2B_2 + c_2C_2$ and $|A| = a_3A_3 + b_3B_3 + c_3C_3$

Note that the sum of the product of elements of any row (or column) with their corresponding cofactors is the value of the determinant.

$$\text{Now } a_1A_2 + b_1B_2 + c_1C_2 = -a_1\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= a_1b_1c_3 + a_1b_3c_1 + a_1b_1c_3 - a_3b_1c_1 - a_1b_3c_1 + a_3b_1c_1 = 0$$

Similarly we get

$$a_1A_3 + b_1B_3 + c_1C_3 = 0 \;;\; a_2A_1 + b_2B_1 + c_2C_1 = 0 \;;$$

$$a_2A_3 + b_2B_3 + c_2C_3 = 0 \;;\; a_3A_1 + b_3B_1 + c_3C_1 = 0 \text{ and } a_3A_2 + b_3B_2 + c_3C_2 = 0.$$

Note 7.12

If elements of a row (or column) are multiplied with corresponding cofactors of any other row (or column) then their sum is zero.

Example 7.31

If A_i, B_i, C_i are the cofactors of a_i, b_i, c_i, respectively, $i = 1$ to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2.$$

Solution

$$\text{Consider the product } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$= \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$$

$$\text{That is, } |A| \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3.$$

$$\Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2.$$

7.3.6 Area of a Triangle

We know that the **area of a triangle** whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is equal to the absolute value of

$$\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3).$$

This expression can be written in the form of a determinant as the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Example 7.32

If the area of the triangle with vertices (– 3, 0), (3, 0) and (0, k) is 9 square units, find the values of k .

Solution

$$\text{Area of the triangle} = \text{absolute value of } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$$9 = \left|\frac{1}{2}\begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}\right| = \left|\frac{1}{2}(-k)(-3-3)\right|$$

$$\Rightarrow \quad 9 = 3|\,k\,| \text{ and hence, } k = \pm 3.$$

Note 7.13

The area of the triangle formed by three points is zero if and only if the three points are collinear. Also, we remind the reader that the determinant could be negative whereas area is always non-negative.

Example 7.33

Find the area of the triangle whose vertices are (– 2, – 3), (3, 2), and (– 1, – 8).

Solution

$$\text{Area of the triangle} = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

$$\left|\frac{1}{2}\begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}(-20+12-22)\right| = |-15| = 15$$

and therefore required area is 15 sq.units.

Example 7.34

Show that the points (a, $b + c$), (b, $c + a$), and (c, $a + b$) are collinear.

Solution

To prove the given points are collinear, it suffices to prove $|A| = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} = 0$.

Applying $C_1 \to C_1 + C_2$, we deduce that

$$|A| = \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = (a+b+c)\times 0 = 0$$

which shows that the given points are collinear.

7.2.11 Singular and non-singular Matrices

Definition 7.21

A square matrix A is said to be **singular** if $|A| = 0$. A square matrix A is said to be **non-singular** if $|A| \neq 0$.

For instance, the matrix $A = \begin{bmatrix} 3 & 8 & 1 \\ -4 & 1 & 1 \\ -4 & 1 & 1 \end{bmatrix}$ is a singular matrix, since

$$|A| = 3(1-1) - 8(-4+4) + 1(-4+4) = 0.$$

If $B = \begin{bmatrix} 2 & 6 & 1 \\ -3 & 0 & 5 \\ 5 & 4 & -7 \end{bmatrix}$ then $|B| = 2(0-20) - (-3)(-42-4) + 5(30-0) = -28 \neq 0.$

Thus B is a non-singular matrix.

Note 7.14

If A and B are non-singular matrices of the same order then AB and BA are also non-singular matrices because $|AB| = |A|\ |B| = |BA|$.

EXERCISE 7.4

(1) Find the area of the triangle whose vertices are (0, 0), (1, 2) and (4, 3).

(2) If (k, 2), (2, 4) and (3, 2) are vertices of the triangle of area 4 square units then determine the value of k.

(3) Identify the singular and non-singular matrices:

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & a-b & k \\ b-a & 0 & 5 \\ -k & -5 & 0 \end{bmatrix}$

(4) Determine the values of a and b so that the following matrices are singular:

(i) $A=\begin{bmatrix}7 & 3\\ -2 & a\end{bmatrix}$ (ii) $B=\begin{bmatrix}b-1 & 2 & 3\\ 3 & 1 & 2\\ 1 & -2 & 4\end{bmatrix}$

(5) If $\cos 2\theta = 0$, determine $\begin{vmatrix}0 & \cos\theta & \sin\theta\\ \cos\theta & \sin\theta & 0\\ \sin\theta & 0 & \cos\theta\end{vmatrix}^2$.

(6) Find the value of the product ; $\begin{vmatrix}\log_3 64 & \log_4 3\\ \log_3 8 & \log_4 9\end{vmatrix} \times \begin{vmatrix}\log_2 3 & \log_8 3\\ \log_3 4 & \log_3 4\end{vmatrix}$.

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EXERCISE 7.5

Choose the correct or the most suitable answer from the given four alternatives.

(1) If $a_{ij}=\frac{1}{2}(3i-2j)$ and $A=[a_{ij}]_{2\times 2}$ is

(1) $\begin{bmatrix}\frac{1}{2} & 2\\ -\frac{1}{2} & 1\end{bmatrix}$ (2) $\begin{bmatrix}\frac{1}{2} & -\frac{1}{2}\\ 2 & 1\end{bmatrix}$ (3) $\begin{bmatrix}2 & 2\\ \frac{1}{2} & -\frac{1}{2}\end{bmatrix}$ (4) $\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ 1 & 2\end{bmatrix}$

(2) What must be the matrix X, if $2X+\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}3 & 8\\ 7 & 2\end{bmatrix}$?

(1) $\begin{bmatrix}1 & 3\\ 2 & -1\end{bmatrix}$ (2) $\begin{bmatrix}1 & -3\\ 2 & -1\end{bmatrix}$ (3) $\begin{bmatrix}2 & 6\\ 4 & -2\end{bmatrix}$ (4) $\begin{bmatrix}2 & -6\\ 4 & -2\end{bmatrix}$

(3) Which one of the following is not true about the matrix $\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 5\end{bmatrix}$?

(1) a scalar matrix (2) a diagonal matrix
(3) an upper triangular matrix (4) a lower triangular matrix

(4) If A and B are two matrices such that $A+B$ and AB are both defined, then
(1) A and B are two matrices not necessarily of same order
(2) A and B are square matrices of same order
(3) Number of columns of A is equal to the number of rows of B
(4) $A = B$.

(5) If $A=\begin{bmatrix}\lambda & 1\\ -1 & -\lambda\end{bmatrix}$, then for what value of λ, $A^2=O$?

(1) 0 (2) ± 1 (3) -1 (4) 1

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(6) If $A=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$, $B=\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$ and $(A+B)^2 = A^2+B^2$, then the values of a and b are

(1) $a=4, b=1$ (2) $a=1, b=4$ (3) $a=0, b=4$ (4) $a=2, b=4$

(7) If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to

(1) (2, – 1) (2) (– 2, 1) (3) (2, 1) (4) (– 2, – 1)

(8) If A is a square matrix, then which of the following is not symmetric?

(1) $A+A^T$ (2) AA^T (3) A^TA (4) $A-A^T$

(9) If A and B are symmetric matrices of order n, where $(A \neq B)$, then

(1) $A + B$ is skew-symmetric (2) $A + B$ is symmetric

(3) $A + B$ is a diagonal matrix (4) $A + B$ is a zero matrix

(10) If $A=\begin{bmatrix}a & x\\ y & a\end{bmatrix}$ and if $xy=1$, then det (AA^T) is equal to

(1) $(a-1)^2$ (2) $(a^2+1)^2$ (3) a^2-1 (4) $(a^2-1)^2$

(11) The value of x, for which the matrix $A=\begin{bmatrix}e^{x-2} & e^{7+x}\\ e^{2+x} & e^{2x+3}\end{bmatrix}$ is singular

(1) 9 (2) 8 (3) 7 (4) 6

(12) If the points $(x,-2)$, $(5,2)$, $(8,8)$ are collinear, then x is equal to

(1) – 3 (2) $\frac{1}{3}$ (3) 1 (4) 3

(13) If $\begin{vmatrix}2a & x_1 & y_1\\ 2b & x_2 & y_2\\ 2c & x_3 & y_3\end{vmatrix} = \frac{abc}{2} \neq 0$, then the area of the triangle whose vertices are $\left(\frac{x_1}{a}, \frac{y_1}{a}\right)$, $\left(\frac{x_2}{b}, \frac{y_2}{b}\right)$, $\left(\frac{x_3}{c}, \frac{y_3}{c}\right)$ is

(1) $\frac{1}{4}$ (2) $\frac{1}{4}abc$ (3) $\frac{1}{8}$ (4) $\frac{1}{8}abc$

(14) If the square of the matrix $\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is the unit matrix of order 2, then α, β and γ should satisfy the relation.

(1) $1+\alpha^2+\beta\gamma=0$ (2) $1-\alpha^2-\beta\gamma=0$

(3) $1-\alpha^2+\beta\gamma=0$ (4) $1+\alpha^2-\beta\gamma=0$

(15) If $\Delta = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$, then $\begin{vmatrix} ka & kb & kc \\ kx & ky & kz \\ kp & kq & kr \end{vmatrix}$ is

(1) Δ (2) $k\Delta$ (3) $3k\Delta$ (4) $k^3\Delta$

(16) A root of the equation $\begin{vmatrix} 3-x & -6 & 3 \\ -6 & 3-x & 3 \\ 3 & 3 & -6-x \end{vmatrix} = 0$ is

(1) 6 (2) 3 (3) 0 (4) – 6

(17) The value of the determinant of $A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ is

(1) – $2abc$ (2) abc (3) 0 (4) $a^2+b^2+c^2$

(18) If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in geometric progression with the same common ratio, then the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are

(1) vertices of an equilateral triangle

(2) vertices of a right angled triangle

(3) vertices of a right angled isosceles triangle

(4) collinear

(19) If $\lfloor . \rfloor$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \le x < 0,\ 0 \le y < 1,\ 1 \le z < 2$, then the value of the determinant $\begin{vmatrix} \lfloor x \rfloor+1 & \lfloor y \rfloor & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor+1 & \lfloor z \rfloor \\ \lfloor x \rfloor & \lfloor y \rfloor & \lfloor z \rfloor+1 \end{vmatrix}$ is

(1) $\lfloor z \rfloor$ (2) $\lfloor y \rfloor$ (3) $\lfloor x \rfloor$ (4) $\lfloor x \rfloor+1$

(20) If $a \ne b, b, c$ satisfy $\begin{vmatrix} a & 2b & 2c \\ 3 & b & c \\ 4 & a & b \end{vmatrix} = 0$, then $abc =$

(1) $a+b+c$ (2) 0 (3) b^3 (4) $ab+bc$

(21) If $A = \begin{vmatrix} -1 & 2 & 4 \\ 3 & 1 & 0 \\ -2 & 4 & 2 \end{vmatrix}$ and $B = \begin{vmatrix} -2 & 4 & 2 \\ 6 & 2 & 0 \\ -2 & 4 & 8 \end{vmatrix}$, then B is given by

(1) $B = 4A$ (2) $B = -4A$ (3) $B = -A$ (4) $B = 6A$

(22) If A is skew-symmetric of order n and C is a column matrix of order $n \times 1$, then $C^T AC$ is

(1) an identity matrix of order n (2) an identity matrix of order 1

(3) a zero matrix of order 1 (4) an identity matrix of order 2

(23) The matrix A satisfying the equation $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is

(1) $\begin{bmatrix} 1 & 4 \\ -1 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 1 & -4 \\ 1 & 0 \end{bmatrix}$ (3) $\begin{bmatrix} 1 & 4 \\ 0 & -1 \end{bmatrix}$ (4) $\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$

(24) If $A + I = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, then $(A+I)(A-I)$ is equal to

(1) $\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$ (2) $\begin{bmatrix} -5 & 4 \\ -8 & 9 \end{bmatrix}$ (3) $\begin{bmatrix} 5 & 4 \\ 8 & 9 \end{bmatrix}$ (4) $\begin{bmatrix} -5 & -4 \\ -8 & -9 \end{bmatrix}$

(25) Let A and B be two symmetric matrices of same order. Then which one of the following statement is not true?

(1) $A + B$ is a symmetric matrix (2) AB is a symmetric matrix

(3) $AB = (BA)^T$ (4) $A^T B = AB^T$

SUMMARY

In this chapter we have acquired the knowledge of

- A matrix is a rectangular array of real numbers or real functions on $\mathbb{R}$ or complex numbers.
- A matrix having m rows and n columns, then the order of the matrix is $m \times n$.
- A matrix $A = [a_{ij}]_{m \times n}$ is said to be a

  square matrix if $m = n$

  row matrix if $m = 1$

  column matrix if $n = 1$

  zero matrix if $a_{ij} = 0 \;\forall\; i$ and j

  diagonal matrix if $m = n$ and $a_{ij} = 0 \;\forall\; i \neq j$

  scalar matrix if $m = n$ and $a_{ij} = 0 \;\forall\; i \neq j$ and $a_{ii} = \lambda$ for all i

  unit matrix or identity matrix if $m = n$ and $a_{ij} = 0$ for all $i \neq j$ and $a_{ii} = 1 \;\forall\; i$

  upper triangular matrix if $m = n$ and $a_{ij} = 0 \;\forall\; i > j$

  lower triangular matrix if $m = n$ and $a_{ij} = 0 \;\forall\; i < j$.
- Matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, are said to be equal if 2iÆ jÆ kÆand iÆ 2jÆ kÆ $\forall\; i$ and j
- If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$
- If $A = [a_{ij}]_{m \times n}$ and λ is a scalar, then $\lambda A = [\lambda a_{ij}]_{m \times n}$

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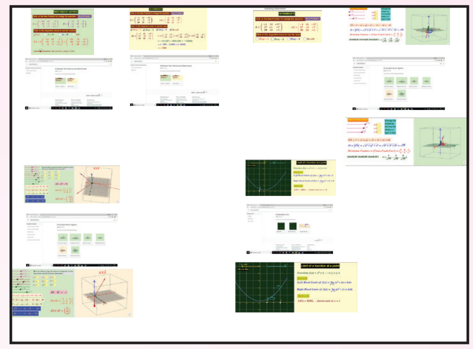
- $-A = (-1)A$
- $A + B = B + A$
- $A - B = A + (-1)B$
- $(A + B) + C = A + (B + C)$ where A, B and C have the same order.
- $A(BC) = (AB)C$ (ii) $A(B+C) = AB+AC$ (iii) $(A+B)C = AC+BC$
- The transpose of A, denoted by A^T is obtained by interchanging rows and columns of A.
  (i) $(A^T)^T = A$, (ii) $(kA)^T = kA^T$, (iii) $(A+B)^T = A^T + B^T$, (iv) $(AB)^T = B^T A^T$
- A square matrix A is called
  (i) symmetric if $A^T = A$ and (ii) skew-symmetric if $A^T = -A$
- Any square matrix can be expressed as sum of a symmetric and skew-symmetric matrices.
- The diagonal entries of a skew-symmetric must be zero.
- For any square matrix A with real entries, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric and further $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.
- Determinant is defined only for square matrices.
- $|A^T| = |A|$.
- $|AB| = |A|\,|B|$ where A and B are square matrices of same order.
- If $A = [a_{ij}]_{m\times n}$, then $|kA| = k^n |A|$, where k is a scalar.
- A determinant of a square matrix A is the sum of products of elements of any row (or column) with its corresponding cofactors; for instance, $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$.
- If the elements of a row or column is multiplied by the cofactors of another row or column, then their sum is zero; for example, $a_{11}A_{13} + a_{12}A_{23} + a_{13}A_{33} = 0$.
- The determinant value remains unchanged if all its rows are interchanged by its columns.
- If all the elements of a row or a column are zero, then the determinant is zero.
- If any two rows or columns are interchanged, then the determinant changes its sign.
- If any two rows or columns are identical or proportional, then the determinant is zero.
- If each element of a row or a column is multiplied by constant k, then determinant gets multiplied by k.
- If each element in any row (column) is the sum of r terms, then the determinant can be expressed as the sum of r determinants.
- A determinant remains unaltered under a row (R_i) operation of the form $R_i + \alpha R_j + \beta R_k (j, k \neq i)$ or a Column (C_i) operation of the form $C_i + \alpha C_j + \beta C_k (j, k \neq i)$ where α, β are scalars.
- **Factor theorem :** If each element of $|A|$ is a polynomial in x and if $|A|$ vanishes for $x = a$, then $x - a$ is a factor of $|A|$.
- Area of the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the absolute value of $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
  If the area is zero, then the three points are collinear.
- A square matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

ICT CORNER 7(a)

Matrices and Determinants

Expected Outcome



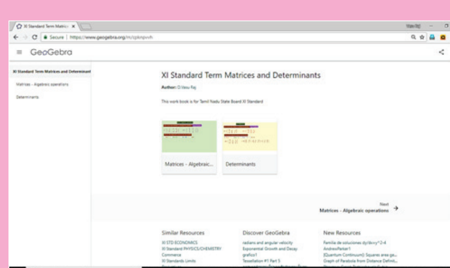
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "Matrices and Determinants" will appear. In that there are several worksheets related to your lesson.

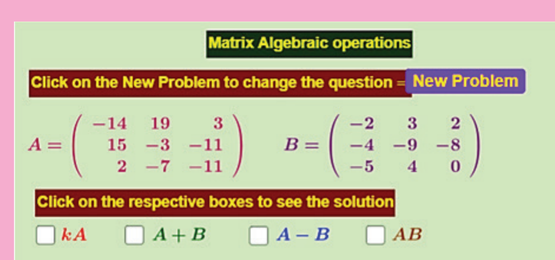
Step 2

Select the work sheet "Matrices-Algebraic operations"work out the operations given and Select the check boxes to verify corresponding answers.

Click on "New Problem" to get new question.



Step1



Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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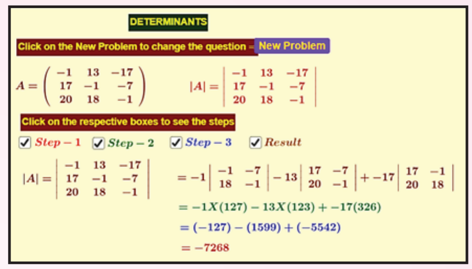
ICT CORNER 7(b)

Matrices and Determinants

Expected Outcome



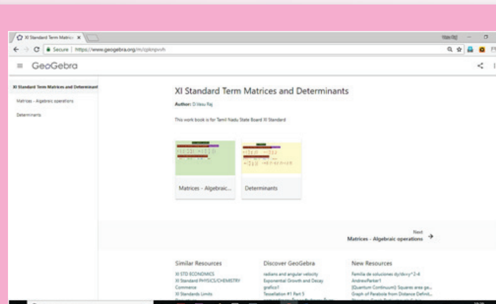
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "Matrices and Determinants" will appear. In that there are several worksheets related to your lesson.

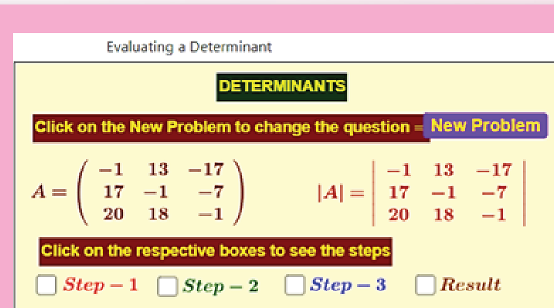
Step 2

Select the work sheet "Determinants" Evaluate the determinant for the matrix given and Select the check boxes to verify steps.

Click on "New Problem" to get new question.



Step1



Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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Chapter 8 Vector Algebra

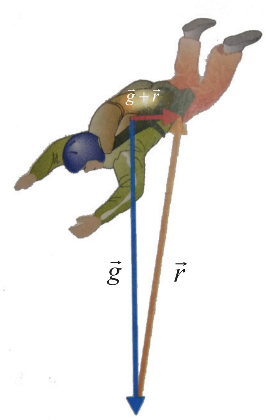
BA7QAF

"On earth there is nothing great but man;
In man there is nothing great but mind"

- Hamilton

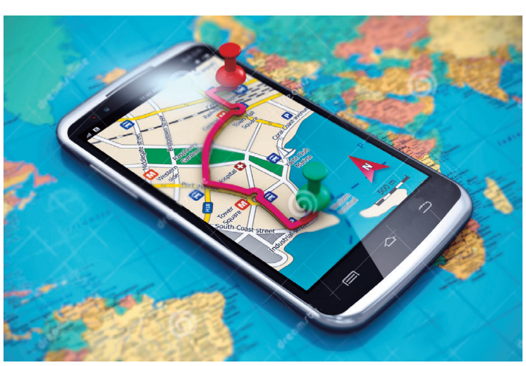
8.1 Introduction

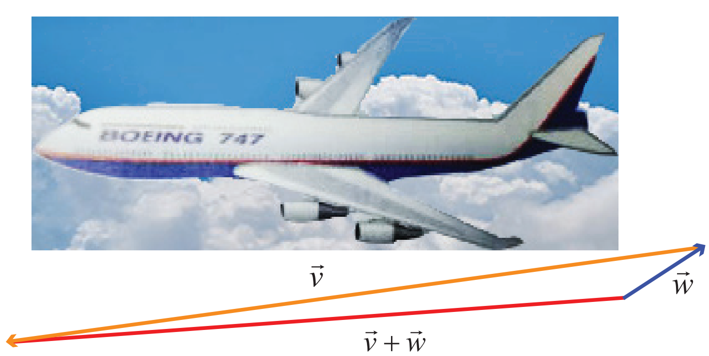
A **pilot constructing a flight plan** has to be concerned about the plane's course, heading, air speed, and ground speed. In order for the plane to proceed directly toward its destination, it must head into the wind at an angle such that the wind is exactly counteracted. If available, a navigation computer will do the calculation quickly and accurately. If, however, a navigation computer is not accessible, the pilot may have to depend on pencil-and-paper work supplemented by a calculator with a knowledge of vectors. An understanding of vectors and their operations is therefore vitally important.



At a certain point during a jump, there are two principal forces acting on a **skydiver**. One force $(\vec{g})$ gravity exerting straight down and another air resistance $(\vec{r})$ exerting up as well as to some direction. What is the net force acting on the skydiver? The answer is $\vec{g}+\vec{r}$. (how?)

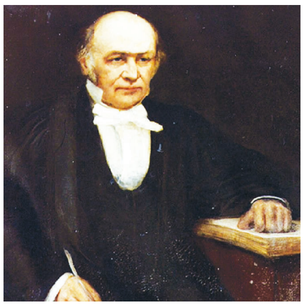
Let $\vec{v}$ be the velocity vector of an aircraft. Suppose that the wind velocity is given by the vector $\vec{w}$, what is the effective velocity of aircraft? The answer is $\vec{v}+\vec{w}$. In what direction should the aircraft head in order to fly due west?





A **global positioning system** (GPS) is a system designed to help to navigate on the earth, in the air and on water. Vectors are also used in GPS.

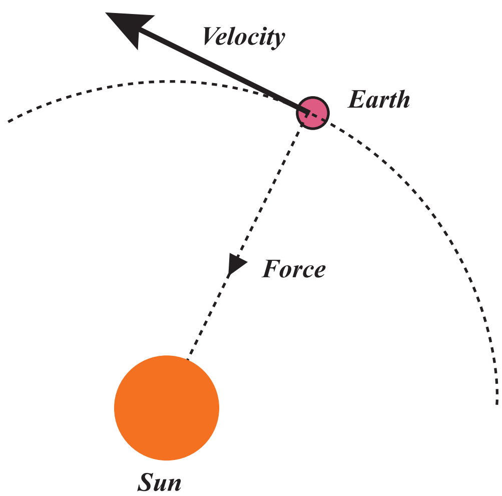
The development of the concept of vectors was influenced by the works of the German mathematician **H.G. Grassmann** (1809 - 1877) and the Irish mathematician **W.R. Hamilton** (1805 – 1865). While Hamilton occupied high positions, Grassman was a secondary school teacher.



Hamilton
(1805 - 1865)

The best features of Quaternion Calculus and Cartesian Geometry were united, largely through the efforts of the American Mathematician **J.B. Gibbs** (1839 - 1903) and **Q.Heaviside** (1850 - 1925) of England and new subject called Vector Algebra was created. The development of the algebra of vectors and of vector analysis as we know it today was first revealed in sets of remarkable notes made by Gibbs for his students at Yale University. **Clifford** (1845 – 1879), in his **Elements of Dynamics** (1878), broke down the product of two quaternions into two very different vector products, which he called the scalar product and the vector product. The term vectors was due to Hamilton and it was derived from the Latin word **'to carry'**.

The theory of vector was also based on Grassman's theory of extension.



It was soon realised that vectors would be the ideal tools for the fruitful study of many ideas in geometry and physics. **Vectors are now the modern language** of a great deal of physics and applied mathematics and they continue to hold their own intrinsic mathematical interest.

Learning Objectives

On completion of this chapter, the students are expected to

- realise vectors as a tool to study the various geometric and physics problems.
- distinguish the scalars from vectors.
- understand different types of vectors and algebra of vectors.
- understand the geometrical interpretations and resolutions of 2D and 3D vectors.
- appreciate the usage of matrix theory in vector algebra.
- visualise scalar product and vector product yielding scalars and vectors respectively as a unique feature.

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8.2 Scalars and Vectors

Definition 8.1

A scalar is a quantity that is determined by its magnitude.

For instance, distance, length, speed, temperature, voltage, mass, pressure, and work are scalars.

Definition 8.2

A **vector** is a quantity that is determined by both its magnitude and its direction and hence it is a directed line segment.

For instance, force, displacement, and velocity (which gives the speed and direction of the motion) are vectors.

We denote vectors by lower case letters with arrow. A two dimensional vector is a directed line segment in $\mathbb{R}^2$ and a three dimensional vector is a directed line segment in $\mathbb{R}^3$.

8.3 Representation of a vector and types of vectors

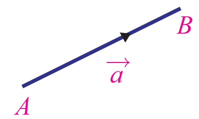

Fig 8.1

A vector has a tail and a tip. Consider the diagram as in Fig. 8.1.

Definition 8.3

The tail point A is called the **initial point** and the tip point B is called the **terminal point** of the vector $\vec{a}$. The initial point of a vector is also taken as origin of the vector.

The initial point A of the vector $\vec{a}$ is the original position of a point and the terminal point B is its position after the translation.

The length or magnitude of the vector $\vec{a}$ is the length of the line segment AB and is denoted by $|\vec{a}|$.

The undirected line AB is called the **support of the vector** $\vec{a}$.
To distinguish between an ordinary line segment without a direction and a line segment representing a vector, we make an arrow mark for the vector as $\overrightarrow{AB}$ and $\vec{a}$. So AB denotes the line segment.

Definition 8.4

If we have a liberty to choose the origin of the vector at any point then it is said to be **free vector,** whereas if it is restricted to a certain specified point then the vector is said to be **localized vector.**

Upto vector product we will be dealing with free vectors only. Localised vectors are involved in finding equations of straight lines.

Definition 8.5

Co-initial vectors are having the same initial point. On the other hand, the **co-terminous vectors** are having the same terminal point.

Definition 8.6

Two or more vectors are said to be **collinear** or **parallel** if they have same line of action or have the lines of action parallel to one another.

Two or more vectors are said to be **coplanar** if they lie on the same plane or parallel to the same plane.

Definition 8.7

Two vectors are said to be **equal** if they have same direction and same magnitude.

Let us note that it is not necessary to have the same initial point and same terminal point for two equal vectors. For instance, in Fig. 8.2, the vectors $\vec{b}$ and $\vec{c}$ are equal since they have same direction and same length, whereas $\vec{a}$ and $\vec{b}$ are not equal because of opposite direction even though they are having same length. The vectors $\vec{c}$ and $\vec{d}$ are not equal even though they are having same direction but not having same length.

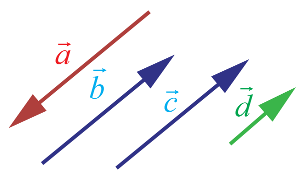

Fig. 8.2

Definition 8.8

Zero vector is a vector which has zero magnitude and an arbitrary direction and it is denoted by $\vec{0}$.

That is, a vector whose initial and terminal points are coincident is called a zero vector.

We observe that the initial and terminal points of a zero vector are the same. The zero vector is also called **null vector** or **void vector**.

A vector of magnitude 1 is called a **unit vector**. The unit vector in the direction of $\vec{a}$ is denoted by $\hat{a}$ (read as 'a cap' or 'a hat'). Clearly $|\hat{a}| = 1$.

We observe that there are infinitely many directions and hence there are infinitely many unit vectors. In fact, for each direction there is one unit vector in that direction.

Any non-zero vector $\vec{a}$ can be written as the scalar multiple of a unit vector in the direction of $\vec{a}$. This scalar is nothing but the magnitude of the vector.

Thus for any vector $\vec{a} = |\vec{a}|\,\hat{a}$, where $\hat{a}$ is the unit vector along the direction of $\vec{a}$.

Clearly $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ for any non-zero vector.

Definition 8.9

Two vectors are said to be **like vectors** if they have the same direction. Two vectors are said to be **unlike vectors** if they have opposite directions.

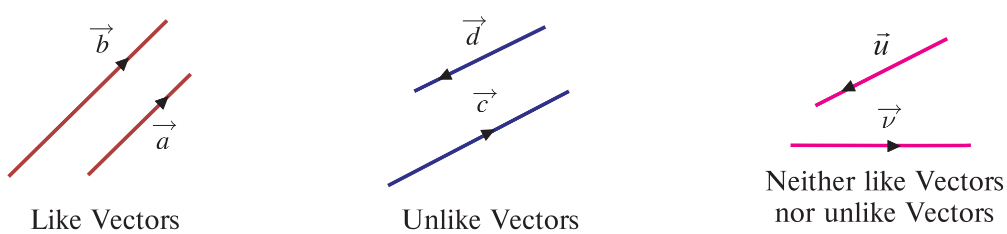

Fig. 8.3

We observe that if two vectors are like vectors or unlike vectors, then the undirected lines (support) of the vectors are parallel to each other. There are pair of vectors which are neither like nor unlike vectors.

8.4 Algebra of Vectors

We have studied basic algebraic operations on real numbers and on matrices. Similarly we studied some operations on vectors. Now let us see how to add two vectors, subtract a vector from another vector and multiply a vector by a scalar.

8.4.1 Addition of Vectors

Let us define the sum of two vectors in two ways and see that they are the same. Let us assume that an object of unit mass is placed at the origin (0,0) in $\mathbb{R}^2$. We assume that the size of the object is just a point. Let us assume that two forces $\vec{a}$ and $\vec{b}$ of unit magnitude act on the object in the positive directions of x-axis and y-axis respectively (Fig.8.4). It is easy to guess that the object will move in the direction 45° to the x-axis as indicated in Fig.8.5. The forces $\vec{a}$ and $\vec{b}$ are equal to the vectors $\vec{a}$ and $\vec{b}$ as indicated in Fig. 8.6. We may think that the forces push the object in Fig. 8.4 and pull the object in Fig.8.6.

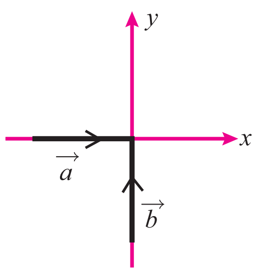

Fig. 8.4

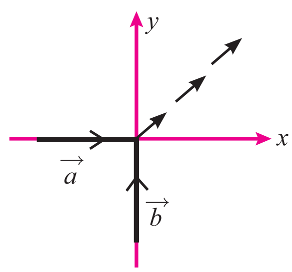

Fig. 8.5

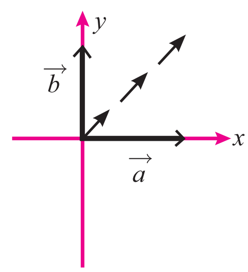

Fig. 8.6

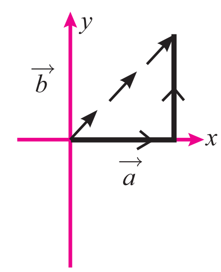

Fig. 8.7

The next question before us is 'How long will it go?'. Let us assume that the forces act one after the other. The force $\vec{a}$ will move the object one unit along the x-axis. So the object will move from (0, 0) to (1, 0). Now the force $\vec{b}$ will move the object vertically from (1, 0) to (1, 1). So finally the object will be at (1, 1) (Fig. 8.7). Thus the sum of the two vectors may be defined as the line segment joining (0, 0) and (1, 1) in the direction '(0, 0) to (1, 1)'.

Now, as in the same situation discussed above, let us assume that the force $\vec{a}$ has magnitude 2 instead of 1 (Fig. 8.8). It will not be difficult to guess that the object will move in a direction much closure to the x-axis as indicated in Fig. 8.9. Also we may guess that the object will go to the point (2,1). Thus the sum of the two vectors may be defined as the line segment joining (0,0) and (2,1) in the direction "(0,0) to (2,1)".

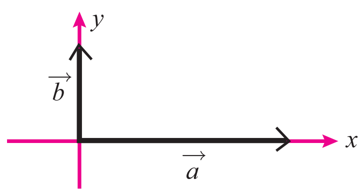

Fig. 8.8

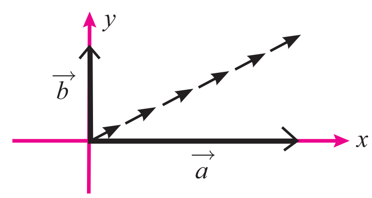

Fig. 8.9

In the two situations discussed above the directions of the forces are perpendicular to each other. This need not be the case in general. Even then we can add the forces by considering one after the other. For example let $\vec{a}$ and $\vec{b}$ be two forces in a plane as shown in Fig. 8.10.

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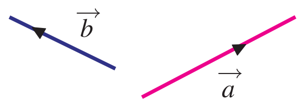


Fig. 8.10

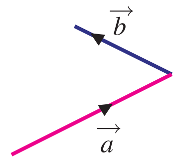


Fig. 8.11

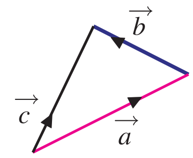


Fig. 8.12

Bringing the initial point of $\vec{b}$ to the terminal point of $\vec{a}$ (Fig. 8.11), we can get the resultant of these two forces (see Fig. 8.12). This motivates us to define the sum of two vectors.

Triangle law of addition

Let $\vec{a}$ and $\vec{b}$ be two vectors. Let A_1 and B_1 be the initial points of $\vec{a}$ and $\vec{b}$, and A_2 and B_2 be the terminal points of $\vec{a}$ and $\vec{b}$ respectively.

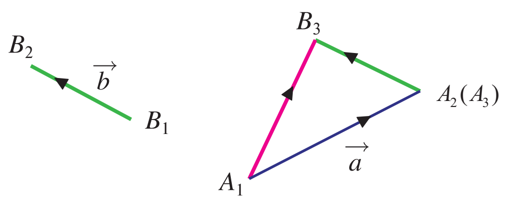


Fig. 8.13

Draw A_3B_3 parallel to B_1B_2 so that $A_3B_3 = B_1B_2$. Then the vector $\overrightarrow{A_1B_3}$ is defined as the sum of the vectors $\vec{a}$ and $\vec{b}$, and it is denoted as $\vec{a}+\vec{b}$. This can be restated as,

Definition 8.10 (Triangle law of addition)

If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum is represented by the third side taken in the reverse order.

Result 8.1

If $\vec{a},\vec{b}$ and $\vec{c}$ are the sides of a triangle taken in order then $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

Proof

Let $\overrightarrow{AB} = \vec{a}, \overrightarrow{BC} = \vec{b}$, and $\overrightarrow{CA} = \vec{c}$.

Now $\vec{a}+\vec{b}+\vec{c} = \overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \overrightarrow{AC}+\overrightarrow{CA} = \overrightarrow{AA} = \vec{0}$.

Thus the result is proved.

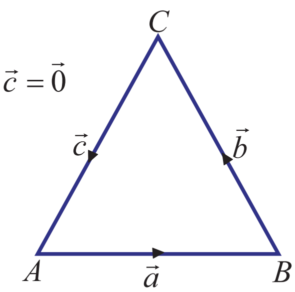


Fig. 8.14

Parallelogram law of vector addition

Let $\vec{a}$ and $\vec{b}$ be two vectors. Assuming that the initial points of the two vectors are the same, let us find the sum according to Definition 8.7. Let A and B be the terminal points of $\vec{a}$ and $\vec{b}$ respectively (Fig. 8.15). To find $\vec{a}+\vec{b}$, we draw AC parallel to OB so that $OB = AC$ and declare that $\overrightarrow{OC}$ is the sum (Fig. 8.16). We observe that OA and BC are parallel (Fig. 8.17).

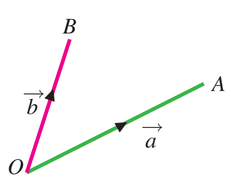


Fig. 8.15

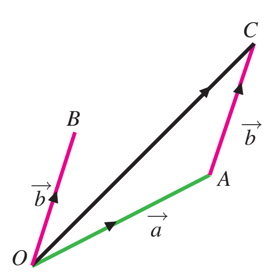


Fig. 8.16

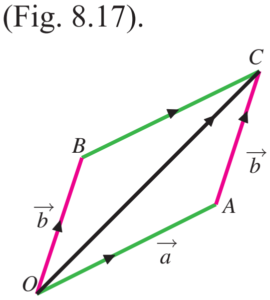


Fig. 8.17

So to find the sum of two vectors with the same initial point, draw the parallelogram with the given vectors as adjacent sides and declare the diagonal as the sum. Even the vectors do not have the same initial point, we can move one of the vectors suitably and make them to have same initial point. This leads us to the following Definition 8.11.

Let $\vec{a}$ and $\vec{b}$ be two vectors with the same initial point O. Let A and B be the terminal points of $\vec{a}$ and $\vec{b}$ respectively.

Complete the parallelogram $OACB$. Then the vector $\overrightarrow{OC}$ is defined as the sum of the vectors $\vec{a}$ and $\vec{b}$. Thus

Definition 8.11 (Parallelogram law of addition)

In a parallelogram $OABC$ if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represents two adjacent sides, then the diagonal $\overrightarrow{OC}$ represents their sum (see Fig. 8.17).

Though we have two definitions for addition of vectors, they are one and the same. Definition 8.10 is defined using the triangle law for addition of vectors and Definition 8.11 is defined using the parallelogram law for addition of vectors:

In a triangle ABC if $\overrightarrow{AB}$ and $\overrightarrow{BC}$ represent two sides, then the third side $\overrightarrow{AC}$ represents their sum.

8.4.2 Difference between two Vectors

Now let us see how to subtract one vector from another vector.

Definition 8.12

Let $\vec{a}$ be a vector. Then the **reverse of** $\vec{a}$, denoted by $-\vec{a}$, is defined as the vector having the magnitude of $\vec{a}$ and the direction opposite to the direction of $\vec{a}$.

Notice that if $\overrightarrow{AB} = \vec{a}$, then $\overrightarrow{BA} = -\vec{a}$.

Geometrical interpretation of difference between two vectors

Let $\vec{a}$ be a vector with initial point P and terminal point Q. Let $\vec{b}$ be the vector with initial point Q and terminal point P. The magnitude of both of the vectors is the length of the line segment joining P and Q. So they have the same magnitude. But clearly they have opposite directions. So $\vec{b}$ is equal to $-\vec{a}$.

If $\vec{a}$ and $\vec{b}$ are two vectors, then the vector $\vec{a}-\vec{b}$ is defined as the sum of the vectors $\vec{a}$ and $-\vec{b}$. That is $\vec{a}+(-\vec{b})$.

We can view the vector $\vec{a}-\vec{b}$ geometrically. Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the vectors $\vec{a}$ and $\vec{b}$ respectively (Fig. 8.18). Draw AC parallel to OB with $AC = OB$. Then $\overrightarrow{AC}$ is equal to $\vec{b}$. Extend the line CA to D so that $CA = AD$. Then $\overrightarrow{AD}$ is equal to $-\vec{b}$. Thus $\vec{a}+(-\vec{b}) = \overrightarrow{OD}$. Hence $\vec{a}-\vec{b} = \overrightarrow{OD}$ (Fig. 8.19).

Let us complete the parallelogram $OACB$. We observe that BA and OD are parallel and they have equal length. Thus the two vectors $\overrightarrow{BA}$ and $\overrightarrow{OD}$ are equal. So we may consider $\overrightarrow{BA}$ as $\vec{a}-\vec{b}$. This shows that if the sides $\overrightarrow{OA}$ and $\overrightarrow{OB}$ of the parallelogram $OACB$ represent the vectors $\vec{a}$ and $\vec{b}$ respectively, then the diagonal $\overrightarrow{BA}$ will represent the vector $\vec{a}-\vec{b}$. (Fig. 8.20). We note that we have already seen that the diagonal $\overrightarrow{OC}$ represents the vector $\vec{a}+\vec{b}$.

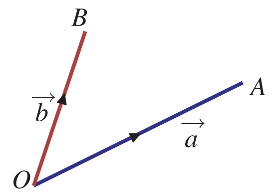


Fig. 8.18

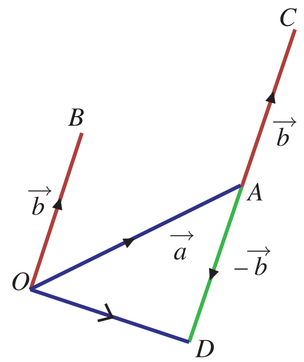


Fig. 8.19

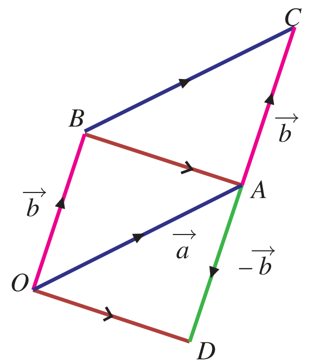


Fig. 8.20

Thus, **if** $\vec{a}$ and $\vec{b}$ **represent two adjacent sides of a parallelogram then the diagonals represent** $\vec{a}+\vec{b}$ **and** $\vec{a}-\vec{b}$**.**

8.4.3 Scalar multiplication of a vector

Now let us see how to multiply a vector by a scalar.

Let $\vec{a}$ be a vector and m be a scalar. Then the vector $m\vec{a}$ is called the **scalar multiple of a vector** $\vec{a}$ by the scalar m.

Let us note that when m is zero, the magnitude of $m\vec{a}$ becomes 0 and hence $m\vec{a}$ becomes the zero vector. If m is positive, then both $\vec{a}$ and $m\vec{a}$ have the same direction and when m is negative, then $\vec{a}$ and $m\vec{a}$ have opposite directions. Thus $\vec{a}$ and $m\vec{a}$ are like vectors if m is positive and unlike vectors if m is negative. The magnitude of $m\vec{a}$ is $|m\vec{a}| = |m||\vec{a}|$.

Definition 8.13

Two vectors $\vec{a}$ and $\vec{b}$ are said to be **parallel** if $\vec{a} = \lambda\vec{b}$, where λ is a scalar. If $\lambda > 0$, they are in the same direction. If $\lambda < 0$ then they are in the opposite direction to each other.

8.4.4 Some properties and results

For any two vectors $\vec{a}$ and $\vec{b}$ and scalars m and n, we have

(i) $m(n\vec{a}) = mn(\vec{a}) = n(m\vec{a})$

(ii) $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(iii) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

BAGLC3

Result 8.2

Vector addition is associative.

For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$,

$$(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c}).$$

Result 8.3

For any vector $\vec{a}$, $\vec{a}+\vec{0} = \vec{0}+\vec{a} = \vec{a}$.

Result 8.4

For any vector $\vec{a}$, $\vec{a}+(-\vec{a}) = (-\vec{a})+\vec{a} = \vec{0}$.

This result states that the additive inverse exists for every vector.

Result 8.5

Vector addition is commutative.

Proof

Let $\vec{a}$ and $\vec{b}$ be two vectors. Let $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$.

Complete the parallelogram $OACB$ with $\vec{a}$ and $\vec{b}$ as adjacent sides. The vectors $\overrightarrow{OB}$ and $\overrightarrow{AC}$ have same direction and equal magnitude; so $\overrightarrow{OB} = \overrightarrow{AC}$. Thus

$$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}.$$

As , $\overrightarrow{OA} = \overrightarrow{BC}$,

$$\vec{b} + \vec{a} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}.$$

Thus $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.

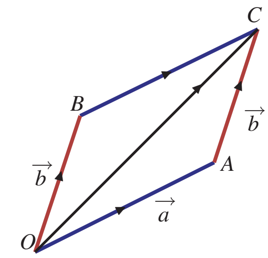


Fig. 8.21

Polygon law of addition

Let $\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD},$ and $\overrightarrow{DE}$ be any five vectors as shown in the Fig. 8.22.

We observe from the figure that each vector is drawn from the terminal point of its previous one. By the triangle law,

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB} \;;\; \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OC}$$

$$\overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OD} \;;\; \overrightarrow{OD} + \overrightarrow{DE} = \overrightarrow{OE}$$

Thus $\overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} = \overrightarrow{OE}$.

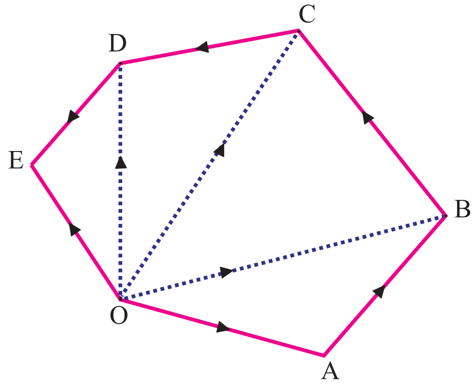


Fig. 8.22

Thus the line joining the initial point of first vector to the terminal point at the last vector is the sum of all the vectors. This is called the **polygon law of addition** of vectors.

Example 8.1

Represent graphically the displacement of

(i) 30 km $60°$ west of north

(ii) 60 km $50°$ south of east.

Solution

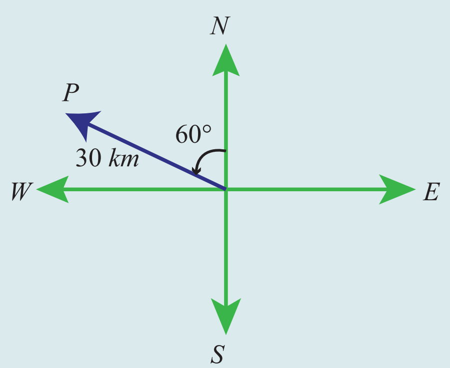


Fig. 8.23

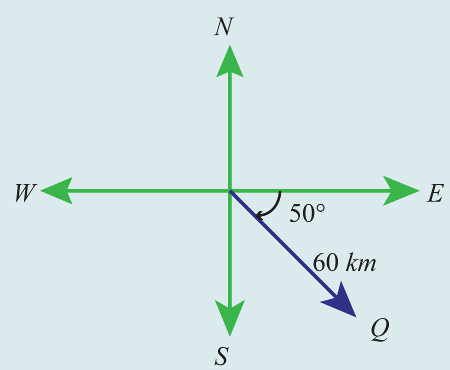


Fig. 8.24

Example 8.2

If $\vec{a}$ and $\vec{b}$ are vectors represented by two adjacent sides of a regular hexagon, then find the vectors represented by other sides.

Solution

Let A,B,C,D,E,F be the vertices of a regular hexagon.

Let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{BC}$

We use the following facts about regular hexagon.

(i) The lines AB, CF and ED are parallel and the lines BC, AD and EF are parallel.

(ii) The length of CF is twice the length of AB and the length of AD is twice the length of BC.

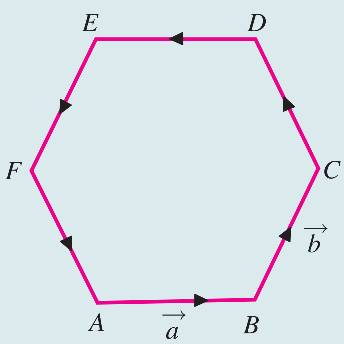


Fig. 8.25

Since the lines AB and DE are parallel, equal in length and opposite in direction we have

$$\overrightarrow{DE} = -\vec{a}.$$

Since the lines AB and CF are parallel and opposite in direction we have

$$\overrightarrow{CF} = -2\vec{a}.$$

Similarly $\overrightarrow{EF} = -\vec{b}$ and $\overrightarrow{AD} = 2\vec{b}$.

Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ we have

$$\overrightarrow{AC} = \vec{a} + \vec{b}.$$

Since $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$ we have

$$\vec{a} + \vec{b} + \overrightarrow{CD} = 2\vec{b}.$$

Thus $\overrightarrow{CD} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}.$

As $\overrightarrow{FA} = -\overrightarrow{CD}$, we have

$$\overrightarrow{FA} = \vec{a} - \vec{b}.$$

Hence, for given sides $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$, we have obtained all other sides of the hexagon as $\overrightarrow{CD} = \vec{b} - \vec{a}, \overrightarrow{DE} = -\vec{a}, \overrightarrow{EF} = -\vec{b}$, and $\overrightarrow{FA} = \vec{a} - \vec{b}$.

8.5 Position vectors

Definition 8.14

Let O be the origin and P be any point (in the plane or space). Then the vector $\overrightarrow{OP}$ is called the **position vector** of the point P with respect to the origin O (point of reference).

The relation between the vectors and position vectors are given in the following result.

Result 8.6

Let O be the origin, A and B be two points. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ where, $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are position vectors of A and B respectively.

Proof

We know that, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$. Thus $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$. ■

Theorem 8.1 (Section Formula - Internal Division)

Let O be the origin. Let A and B be two points. Let P be the point which divides the line segment AB internally in the ratio $m : n$. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, then the position vector $\overrightarrow{OP}$ of P is given by

$$\overrightarrow{OP} = \frac{n\vec{a}+m\vec{b}}{n+m}.$$

Proof

Since O is the origin, $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, we have

$$\overrightarrow{OA} = \vec{a} \text{ and } \overrightarrow{OB} = \vec{b}.$$

$$\text{Let } \overrightarrow{OP} = \vec{r}.$$

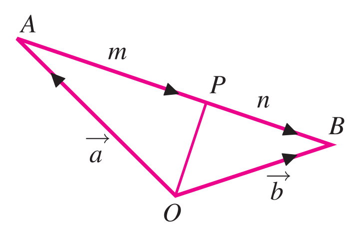


Fig. 8.26

Since P divides the line segment AB internally in the ratio $m : n$, we have,

$$\frac{|\overrightarrow{AP}|}{|\overrightarrow{PB}|} = \frac{m}{n}$$

and hence $$n|\overrightarrow{AP}| = m|\overrightarrow{PB}|.$$

But the vectors $\overrightarrow{AP}$ and $\overrightarrow{PB}$ have the same direction. Thus

$$n\overrightarrow{AP} = m\overrightarrow{PB}. \quad (8.1)$$

$$\text{But } \overrightarrow{AP} = \overrightarrow{OP}-\overrightarrow{OA} = \vec{r}-\vec{a} \text{ and } \overrightarrow{PB} = \overrightarrow{OB}-\overrightarrow{OP} = \vec{b}-\vec{r}$$

Substituting this in (8.1), we get

$$n(\vec{r}-\vec{a}) = m(\vec{b}-\vec{r})$$

and hence

$$(n+m)\vec{r} = n\vec{a}+m\vec{b}.$$

Thus $$\overrightarrow{OP} = \frac{n\vec{a}+m\vec{b}}{n+m}.$$ ■

Theorem 8.2 Section Formula - External Division (Without proof)

Let O be the origin. Let A and B be two points. Let P be the point which divides the line segment AB externally in the ratio $m : n$. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, then the position vector $\overrightarrow{OP}$ of P is given by

$$\overrightarrow{OP} = \frac{n\vec{a}-m\vec{b}}{n-m}.$$

Note 8.1

By taking $m = n = 1$ in Theorem 8.1, we see that the position vector of the midpoint of the line joining the points A and B is $\frac{\vec{a}+\vec{b}}{2}$, where $\vec{a}$ and $\vec{b}$ are the position vectors of the points A and B respectively.

From the above theorem we can get a condition for three points to be collinear.

Result 8.7

Three distinct points *A, B* and *C* with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if and only if there exist real numbers *x,y,z,* none of them is zero, such that

$$x+y+z=0 \text{ and } x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}.$$

Example 8.3

Let A and B be two points with position vectors $2\vec{a}+4\vec{b}$ and $2\vec{a}-8\vec{b}$. Find the position vectors of the points which divide the line segment joining *A* and *B* in the ratio 1:3 internally and externally.

Solution

Let O be the origin. It is given that

$$\overrightarrow{OA}=2\vec{a}+4\vec{b} \text{ and } \overrightarrow{OB}=2\vec{a}-8\vec{b}.$$

Let C and D be the points which divide the segment AB in the ratio 1 : 3 internally and externally respectively. Then

$$\overrightarrow{OC}=\frac{3\overrightarrow{OA}+\overrightarrow{OB}}{3+1}=\frac{3(2\vec{a}+4\vec{b})+(2\vec{a}-8\vec{b})}{4}=2\vec{a}+\vec{b}.$$

$$\overrightarrow{OD}=\frac{3\overrightarrow{OA}-\overrightarrow{OB}}{3-1}=\frac{3(2\vec{a}+4\vec{b})-(2\vec{a}-8\vec{b})}{2}=2\vec{a}+10\vec{b}.$$

Let us recall the definition that the line joining a vertex of a triangle with the midpoint of its opposite side is called a median. The centroid divides the median from vertex to the midpoint of the opposite side internally in the ratio 2:1.

Theorem 8.3

The medians of a triangle are concurrent.

Proof

Let *ABC* be a triangle and let *D, E, F* be the mid points of its sides *BC*, *CA* and *AB* respectively. We have to prove that the medians *AD, BE, CF* are concurrent.

Let O be the origin and $\vec{a},\vec{b},\vec{c}$ be the position vectors of *A, B,* and *C* respectively.

The position vectors of *D, E,* and *F* are respectively

$$\frac{\vec{b}+\vec{c}}{2},\frac{\vec{c}+\vec{a}}{2},\frac{\vec{a}+\vec{b}}{2}.$$

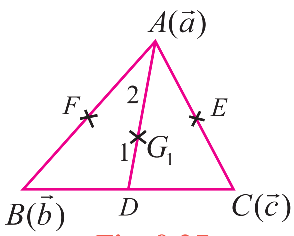


Fig. 8.27

Let G_1 be the point on AD dividing it internally in the ratio 2 : 1

Therefore, position vector of $G_1=\dfrac{1\overrightarrow{OA}+2\overrightarrow{OD}}{1+2}$

$$\overrightarrow{OG_1}=\frac{1\vec{a}+2\left(\frac{\vec{b}+\vec{c}}{2}\right)}{3}=\frac{\vec{a}+\vec{b}+\vec{c}}{3} \qquad (1)$$

Let G_2 be the point on BE dividing it internally in the ratio 2 : 1

Therefore, $\overrightarrow{OG_2}=\dfrac{1\overrightarrow{OB}+2\overrightarrow{OE}}{1+2}$

$$\overrightarrow{OG_2} = \frac{1\vec{b} + 2\left(\frac{\vec{c} + \vec{a}}{2}\right)}{3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3}. \qquad (2)$$

Similarly if G_3 divides CF in the ratio 2 : 1 then

$$\overrightarrow{OG_3} = \frac{\vec{a} + \vec{b} + \vec{c}}{3} \qquad (3)$$

From (1), (2), and (3) we find that the position vectors of the three points G_1, G_2, G_3 are one and the same. Hence they are not different points. Let the common point be G.

Therefore the three medians are concurrent and the point of concurrence is G.

Theorem 8.4

A quadrilateral is a parallelogram if and only if its diagonals bisect each other.

Proof

Let A, B, C, D be the vertices of a quadrilateral with diagonals AC and BD. Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be the position vectors of A, B, C, and D respectively with respect to O.

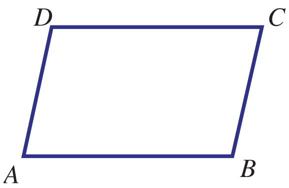


Fig. 8.28

Let the quadrilateral $ABCD$ be a parallelogram. Then

$$\overrightarrow{AB} = \overrightarrow{DC} \Rightarrow \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD} \Rightarrow \vec{b} - \vec{a} = \vec{c} - \vec{d} \Rightarrow \vec{b} + \vec{d} = \vec{a} + \vec{c}$$

and hence $$\frac{\vec{b} + \vec{d}}{2} = \frac{\vec{a} + \vec{c}}{2}.$$

This shows that the position vectors of the midpoint of the line segments AC and BD are the same. In other words, the diagonals bisect each other.

Conversely let us assume that the diagonal bisects each other. Thus the position vectors of the midpoint of the line segments AC and BD are the same. Thus

$$\frac{\vec{a} + \vec{c}}{2} = \frac{\vec{b} + \vec{d}}{2} \Rightarrow \vec{a} + \vec{c} = \vec{b} + \vec{d} \Rightarrow \vec{c} - \vec{d} = \vec{b} - \vec{a}.$$

This implies that $\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$ and hence $\overrightarrow{DC} = \overrightarrow{AB}$. This shows that the lines AB and DC are parallel. From $\vec{a} + \vec{c} = \vec{b} + \vec{d}$ we see that $\vec{a} - \vec{d} = \vec{b} - \vec{c}$ which shows that the lines AD and BC are parallel. Hence $ABCD$ is a parallelogram.

EXERCISE 8.1

(1) Represent graphically the displacement of

(i) $45cm\ 30°$ north of east. (ii) $80km,\ 60°$ south of west

(2) Prove that the relation R defined on the set V of all vectors by '$\vec{a}\ R\ \vec{b}$ if $\vec{a} = \vec{b}$' is an equivalence relation on V.

(3) Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A and B. Prove that the position vectors of the points which trisects the line segment AB are $\frac{\vec{a} + 2\vec{b}}{3}$ and $\frac{\vec{b} + 2\vec{a}}{3}$.

(4) If D and E are the midpoints of the sides AB and AC of a triangle ABC, prove that $\overrightarrow{BE}+\overrightarrow{DC}=\frac{3}{2}\overrightarrow{BC}$.

(5) Prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side whose length is half of the length of the third side.

(6) Prove that the line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram.

(7) If $\vec{a}$ and $\vec{b}$ represent a side and a diagonal of a parallelogram, find the other sides and the other diagonal.

(8) If $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$, prove that the points P, Q, R are collinear.

(9) If D is the midpoint of the side BC of a triangle ABC, prove that $\overrightarrow{AB}+\overrightarrow{AC}=2\overrightarrow{AD}$.

(10) If G is the centroid of a triangle ABC, prove that $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\vec{0}$.

(11) Let A, B, and C be the vertices of a triangle. Let D,E, and F be the midpoints of the sides BC, CA, and AB respectively. Show that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$.

(12) If $ABCD$ is a quadrilateral and E and F are the midpoints of AC and BD respectively, then prove that $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}=4\overrightarrow{EF}$.

8.6 Resolution of Vectors

Resolution of a vector can be done for any finite dimension. But we will discuss only in two and three dimensions. Let us start with two dimension.

8.6.1 Resolution of a vector in two dimension

Theorem 8.5

Let $\hat{i}$ and $\hat{j}$ be the unit vectors along the positive x-axis and the y-axis having initial point at the origin O. Now $\overrightarrow{OP}$ is the position vector of any point P in the plane. Then $\overrightarrow{OP}$ can be uniquely written as

$\overrightarrow{OP} = x\hat{i}+y\hat{j}$ for some real numbers x and y. Further $|\overrightarrow{OP}| = \sqrt{x^2+y^2}$.

Proof

Let (x, y) be the coordinates of the point P. Let L and M be the foots of the perpendiculars drawn from P to the x and y axes. Then $\overrightarrow{OP}=\overrightarrow{OL}+\overrightarrow{LP}=\overrightarrow{OL}+\overrightarrow{OM}$.

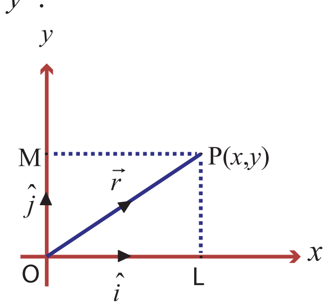


Fig. 8.29

Since $\hat{i}$ and $\hat{j}$ are unit vectors, we have $\overrightarrow{OL}=x\hat{i}$ and $\overrightarrow{OM}=y\hat{j}$.

Thus $\overrightarrow{OP} = x\hat{i}+y\hat{j}$

If $\overrightarrow{OP}=\vec{r}$ then $\vec{r}=x\hat{i}+y\hat{j}$.

To prove the uniqueness, let $x_1\hat{i}+y_1\hat{j}$ and $x_2\hat{i}+y_2\hat{j}$ be two representations of the same point P. Then

$$x_1\hat{i}+y_1\hat{j} = x_2\hat{i}+y_2\hat{j}.$$

This implies that $(x_1-x_2)\hat{i}-(y_2-y_1)\hat{j} = \vec{0} \Rightarrow x_1-x_2=0, y_2-y_1=0$.

In other words $x_1=x_2$ and $y_1=y_2$ and hence the uniqueness follows.

In the triangle OLP, $OP^2 = OL^2 + LP^2$; hence $|\overrightarrow{OP}| = \sqrt{x^2+y^2}$.

That is, $|\vec{r}| = r = \sqrt{x^2+y^2}$.

Observe that if $\hat{i}$ and $\hat{j}$ are the unit vectors in the postive directions of x and y axes, then the position vector of the point (6,4) can be written as $6\hat{i}+4\hat{j}$ and this is the only way of writing it.

Result 8.8

If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors in a plane, then any vector in the plane can be written as the linear combination of $\vec{a}$ and $\vec{b}$ in a unique way. That is, any vector in the plane is of the form $l\vec{a}+m\vec{b}$ for some scalars l and m.

Proof

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, and $\vec{r}$ be any vector coplanar with $\vec{a}$ and $\vec{b}$.

Draw PL parallel to OB. Clearly $\overrightarrow{LP} = m\vec{b}$ and $\overrightarrow{OL} = l\vec{a}$ for some l and m.

Now $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP}$.

That is, $\vec{r} = l\vec{a} + m\vec{b}$.

Therefore if $\vec{r}, \vec{a}, \vec{b}$ are coplanar then $\vec{r}$ is a linear combination of $\vec{a}$ and $\vec{b}$.

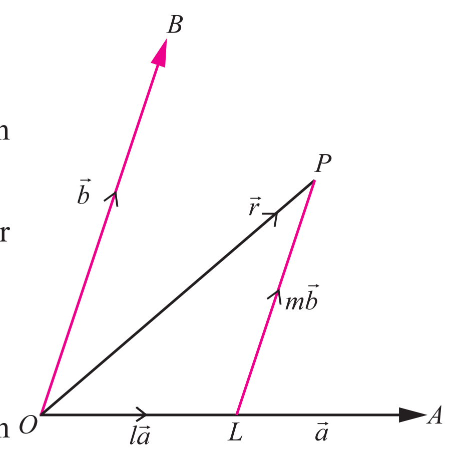


Fig. 8.30

Note 8.2

Further if three non collinear vectors are coplanar then any one of the vector can be written as a linear combination of other two. Note that the converse is also true.

Result 8.9

If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors in the space, then any vector in the space can be written as $l\vec{a} + m\vec{b} + n\vec{c}$ in a unique way for some scalars l, m and n.

Definition 8.15

Let $\hat{i}$ and $\hat{j}$ be the unit vectors in the positive directions of x and y axes respectively. Let $\vec{r}$ be any vector in the plane. Then $\vec{r} = x\hat{i} + y\hat{j}$ for some real numbers x and y. Here $x\hat{i}$ and $y\hat{j}$ are called the **rectangular components** of $\vec{r}$ along the x and y axes respectively in two dimension.

What we discussed so far can be discussed in the three dimensional space also.

8.6.2 Resolution of a vector in three dimension

Theorem 8.6

Let $\hat{i}$, $\hat{j}$ and $\hat{k}$ be the unit vectors in the direction of postive x, y and z axes respectively having initial point at the origin O. Let $\overrightarrow{OP}$ be the position vector of any point P in the space. Then $\overrightarrow{OP}$ can be uniquely written as $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$ for some real numbers x, y and z. Further $|\overrightarrow{OP}| = \sqrt{x^2+y^2+z^2}$.

Proof

Let (x, y, z) be the coordinates of the point P. Let Q be the foot of the perpendicular drawn from P to the xy-plane. Let R and S be the foots of the perpendiculars drawn from Q to the x and y axes respectively. Let $\overrightarrow{OP} = \vec{r}$.

Then, $OR = x$, $OS = y$, and $QP = z$.

Thus, $\overrightarrow{OR} = x\hat{i}$, $\overrightarrow{RQ} = \overrightarrow{OS} = y\hat{j}$, and $\overrightarrow{QP} = z\hat{k}$

$$\overrightarrow{OP} = \vec{r} = \overrightarrow{OQ} + \overrightarrow{QP} = \overrightarrow{OR} + \overrightarrow{RQ} + \overrightarrow{QP} = x\hat{i} + y\hat{j} + z\hat{k}.$$

That is $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

This $\overrightarrow{OP}$ vector is called the position vector of P with respect to the origin O in three dimension.

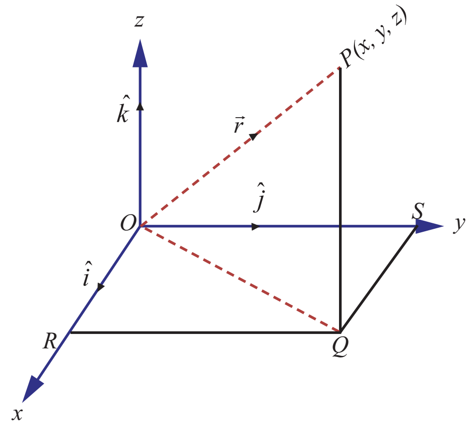


Fig. 8.31

In the triangle ORQ,

$$OQ^2 = OR^2 + RQ^2 \text{ (how?)}$$

and in the triangle OQP,

$$OP^2 = OQ^2 + QP^2.$$

Thus $OP^2 = OQ^2 + QP^2 = OR^2 + RQ^2 + QP^2 = x^2 + y^2 + z^2$

and hence $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$, that is $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$ ■

Components of vector joining two points

Let us find the components of the vector joining the point (x_1, y_1) to (x_2, y_2).

Let A and B be the points (x_1, y_1) and (x_2, y_2). Let P be the point $(x_2 - x_1, y_2 - y_1)$. Then $\overrightarrow{AB} = \overrightarrow{OP}$. The components of $\overrightarrow{OP}$ are $(x_2 - x_1)\hat{i}$ and $(y_2 - y_1)\hat{j}$. Hence the components of $\overrightarrow{AB}$ in the directions of x and y axes are $(x_2 - x_1)\hat{i}$ and $(y_2 - y_1)\hat{j}$.

Similarly if A and B are the points (x_1, y_1, z_1) and (x_2, y_2, z_2), then the components of $\overrightarrow{AB}$ in the directions of x, y and z axes are $(x_2 - x_1)\hat{i}$, $(y_2 - y_1)\hat{j}$, and $(z_2 - z_1)\hat{k}$.

8.6.3 Matrix representation of a vector

A vector with three components can be visualised as either a row or column matrix as $[x, y, z]$ or $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ respectively.

Thus any vector $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ can be obtained from $[a_1 \;\; a_2 \;\; a_3]\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{A}$.

Hence addition of vectors and multiplication of a vector by a scalar can be defined as follows.

If $\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{A} + \vec{B} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}$ resulting in $\vec{A} + \vec{B} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$.

Also $k\vec{A} = k\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ka_1 \\ ka_2 \\ ka_3 \end{bmatrix}$ yielding

$k\vec{A} = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$

For $k \in \mathbb{R}, k > 1$ yields magnification, $0 < k < 1$ yields contraction of a vector and $k = 0$ yields a zero vector $\overline{OA} = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$.

Result 8.10

Using the commutative, associative properties of vector addition and the distributive property of the scalar multiplication we can prove the following.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and if m is a scalar, then

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

(iii) $m\vec{a} = ma_1\hat{i} + ma_2\hat{j} + ma_3\hat{k}$ and

(iv) $\vec{a} = \vec{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Example 8.4

Find a unit vector along the direction of the vector $5\hat{i} - 3\hat{j} + 4\hat{k}$.

Solution

We know that a unit vector along the direction of the vector $\vec{a}$ is given by $\dfrac{\vec{a}}{|\vec{a}|}$. So a unit vector along the direction of $5\hat{i} - 3\hat{j} + 4\hat{k}$ is given by

$$\frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{|5\hat{i} - 3\hat{j} + 4\hat{k}|} = \frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{5^2 + 3^2 + 4^2}} = \frac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{50}}.$$

Note 8.3

Now we have another unit vector parallel to $5\hat{i} - 3\hat{j} + 4\hat{k}$ in the opposite direction. That is, $-\dfrac{5\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{50}}$.

8.7 Direction Cosines and Direction Ratios

Let P be a point in the space with coordinates (x, y, z) and of distance r from the origin. Let R, S and T be the foots of the perpendiculars drawn from P to the x, y and z axes respectively. Then

$$\angle PRO = \angle PSO = \angle PTO = 90^\circ.$$
$$OR = x,\ OS = y,\ OT = z \text{ and } OP = r.$$

(It may be difficult to visualize that $\angle PRO = \angle PSO = \angle PTO = 90^\circ$ in the figure; as they are foot of the perpendiculars to the axes from P; in a three dimensional model we can easily visualize the fact.)

Let α, β, γ be the angles made by the vector $\overrightarrow{OP}$ with the positive x, y and z axes respectively. That is,

$$\angle POR = \alpha,\quad \angle POS = \beta,\quad \text{and}\quad \angle POT = \gamma.$$

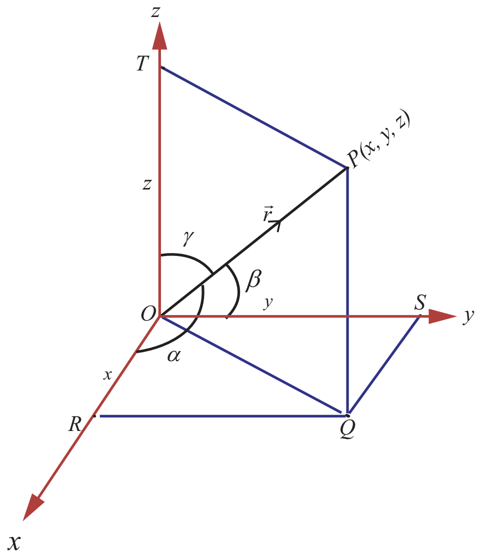


Fig. 8.32

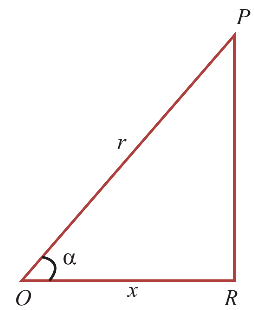


Fig. 8.33

In $\Delta OPR, \angle PRO = 90°, \angle POR = \alpha, OR = x,$ and $OP = r.$ Therefore

$$\cos\alpha = \frac{OR}{OP} = \frac{x}{r}.$$

In a similar way we can find that $\cos\beta = \frac{y}{r}$ and $\cos\gamma = \frac{z}{r}$.

Here the angles α, β, γ are called direction angles of the vector $\overrightarrow{OP} = \vec{r}$ and $\cos\alpha, \cos\beta, \cos\gamma$ are called **direction cosines** of the vector $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$. Thus $\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$, where $r = \sqrt{x^2 + y^2 + z^2}$, are the direction cosines of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Any three numbers which are proportional to the direction cosines of vector are called the **direction ratios** of the vector. Hence the direction ratios of a vector is not unique. For a given vector, we have, infinitely many set of direction ratios.

Observations

(i) For a given non-zero vector, one can find the direction ratios as well as the direction cosines.

(ii) For a given set of direction ratios, one cannot find the corresponding vector.

(iii) For a given set of direction cosines, one cannot find the corresponding vector.

(iv) For a given vector, the triplet of direction cosines is also a triplet of direction ratios.

(v) To find the vector, the magnitude as well as either the set of direction cosines or a set of direction ratios are essential.

Note 8.4

Here we consider a vector $\overrightarrow{OP}$ whose initial point is at the origin. If the vector whose initial point is not the origin, then, in order to find its direction cosines, we draw a vector with initial point at the origin and parallel to the given vector of same magnitude by translation. By the principle of two equal vectors having the same set of direction cosines, we can find direction cosines of any vector.

Result 8.11

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point and let α, β, γ be the direction angles of $\vec{r}$. Then

(i) the sum of the squares of the direction cosines of $\vec{r}$ is 1.

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(iii) the direction cosines of $\vec{r}$ are $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$.

(iv) l, m, n are the direction cosines of a vector if and only if $l^2+m^2+n^2=1$.

(v) any unit vector can be written as $\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k}$

Proof

(i) $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$

The proofs of (ii), (iii), (iv), and (v) are left as exercise.

Example 8.5

Find a direction ratio and direction cosines of the following vectors.

(i) $3\hat{i}+4\hat{j}-6\hat{k}$, (ii) $3\hat{i}-4\hat{k}$.

Solution

(i) The direction ratios of $3\hat{i}+4\hat{j}-6\hat{k}$ are 3, 4, – 6.

The direction cosines are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, where $r=\sqrt{x^2+y^2+z^2}$.

Therefore, the direction cosines are $\frac{3}{\sqrt{61}}, \frac{4}{\sqrt{61}}, \frac{-6}{\sqrt{61}}$

(ii) The direction ratios of $3\hat{i}-4\hat{k}$ are 3, 0, – 4.

The direction cosines are $\frac{3}{5}, 0, \frac{-4}{5}$.

Example 8.6

(i) Find the direction cosines of a vector whose direction ratios are 2, 3, – 6.

(ii) Can a vector have direction angles $30°, 45°, 60°$?

(iii) Find the direction cosines of $\overrightarrow{AB}$, where A is (2, 3, 1) and B is (3, – 1, 2).

(iv) Find the direction cosines of the line joining (2, 3, 1) and (3, – 1, 2).

(v) The direction ratios of a vector are 2, 3, 6 and it's magnitude is 5. Find the vector.

Solution

(i) The direction cosines are $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$.

That is, $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$.

(ii) The condition is $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Here $\alpha=30°, \beta=45°, \gamma=60°$

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \frac{3}{4}+\frac{1}{2}+\frac{1}{4} \neq 1$.

Therefore they are not direction angles of any vector.

(iii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} - 4\hat{j} + \hat{k}$

Direction cosines are $\frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}$.

(iv) Let A and B be the points (2, 3, 1) and (3,–1, 2). The direction cosines of $\overrightarrow{AB}$ are $\frac{1}{\sqrt{18}}, \frac{-4}{\sqrt{18}}, \frac{1}{\sqrt{18}}$. But any point can be taken as first point. Hence we have another set of direction cosines with opposite direction. Thus, we have another set of direction ratios $\frac{-1}{\sqrt{18}}, \frac{4}{\sqrt{18}}, \frac{-1}{\sqrt{18}}$.

(v) The direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$.

The unit vector is $\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$.

The required vector is $\frac{5}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Example 8.7

Show that the points whose position vectors are $2\hat{i} + 3\hat{j} - 5\hat{k}$, $3\hat{i} + \hat{j} - 2\hat{k}$ and , $6\hat{i} - 5\hat{j} + 7\hat{k}$ are collinear.

Solution

Let O be the origin and let $\overrightarrow{OA}, \overrightarrow{OB}$, and $\overrightarrow{OC}$ be the vectors $2\hat{i} + 3\hat{j} - 5\hat{k}$, $3\hat{i} + \hat{j} - 2\hat{k}$ and $6\hat{i} - 5\hat{j} + 7\hat{k}$ respectively. Then

$$\overrightarrow{AB} = \hat{i} - 2\hat{j} + 3\hat{k} \text{ and } \overrightarrow{AC} = 4\hat{i} - 8\hat{j} + 12\hat{k}.$$

Thus $\overrightarrow{AC} = 4\overrightarrow{AB}$ and hence $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel. They have a common point namely A. Thus, the three points are collinear.

Alternative method

Let O be the point of reference.

Let $\overrightarrow{OA} = 2\hat{i} + 3\hat{j} - 5\hat{k}, \overrightarrow{OB} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{OC} = 6\hat{i} - 5\hat{j} + 7\hat{k}$

$\overrightarrow{AB} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\overrightarrow{BC} = 3\hat{i} - 6\hat{j} + 9\hat{k}$; $\overrightarrow{CA} = -4\hat{i} + 8\hat{j} - 12\hat{k}$

$|\overrightarrow{AB}| = \sqrt{14}$; $|\overrightarrow{BC}| = \sqrt{126} = 3\sqrt{14}$; $|\overrightarrow{CA}| = \sqrt{224} = 4\sqrt{14}$.

Thus, $AC = AB + BC$.

Hence A, B, C are lying on the same line. That is, they are collinear.

Example 8.8

Find a point whose position vector has magnitude 5 and parallel to the vector $4\hat{i} - 3\hat{j} + 10\hat{k}$.

Solution :

Let $\vec{a}$ be the vector $4\hat{i} - 3\hat{j} + 10\hat{k}$.

The unit vector $\hat{a}$ along the direction of $\vec{a}$ is $\frac{\vec{a}}{|\vec{a}|}$ which is equal to $\frac{4\hat{i}-3\hat{j}+10\hat{k}}{5\sqrt{5}}$. The vector whose magnitude is 5 and parallel to $4\hat{i}-3\hat{j}+10\hat{k}$, is $5\left(\frac{4\hat{i}-3\hat{j}+10\hat{k}}{5\sqrt{5}}\right)=\frac{4\hat{i}}{\sqrt{5}}-\frac{3}{\sqrt{5}}\hat{j}+2\sqrt{5}\;\hat{k}$.

So a required point is $\left(\frac{4}{\sqrt{5}},-\frac{3}{\sqrt{5}},2\sqrt{5}\right)$.

Example 8:9

Prove that the points whose position vectors $2\hat{i}+4\hat{j}+3\hat{k}$, $4\hat{i}+\hat{j}+9\hat{k}$ and $10\hat{i}-\hat{j}+6\hat{k}$ form a right angled triangle.

Solution

Let A, B, C be the given points and O be the point of reference or origin.

Then $\overrightarrow{OA} = 2\hat{i}+4\hat{j}+3\hat{k}$, $\overrightarrow{OB}=4\hat{i}+\hat{j}+9\hat{k}$ and $\overrightarrow{OC}=10\hat{i}-\hat{j}+6\hat{k}$

$$\overrightarrow{AB} = \overrightarrow{OB}-\overrightarrow{OA}=(4\hat{i}+\hat{j}+9\hat{k})-(2\hat{i}+4\hat{j}+3\hat{k})=2\hat{i}-3\hat{j}+6\hat{k}.$$

$$AB = |\overrightarrow{AB}| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{4+9+36}=7$$

$$\overrightarrow{BC} = \overrightarrow{OC}-\overrightarrow{OB}=(10\hat{i}-\hat{j}+6\hat{k})-(4\hat{i}+\hat{j}+9\hat{k})=6\hat{i}-2\hat{j}-3\hat{k}.$$

$$BC = |\overrightarrow{BC}| = \sqrt{6^2+(-2)^2+(-3)^2} = \sqrt{36+4+9}=7$$

$$\overrightarrow{CA} = \overrightarrow{OA}-\overrightarrow{OC}=(2\hat{i}+4\hat{j}+3\hat{k})-(10\hat{i}-\hat{j}+6\hat{k})=-8\hat{i}+5\hat{j}-3\hat{k}.$$

$$CA = |\overrightarrow{CA}| = \sqrt{(-8)^2+5^2+(-3)^2} = \sqrt{64+25+9}=\sqrt{98}$$

$$BC^2 = 49,\ CA^2=98,\ AB^2=49.$$

Clearly $CA^2 = BC^2+AB^2$.

Therefore, the given points form a right angled triangle.

Example 8.10

Show that the vectors $5\hat{i}+6\hat{j}+7\hat{k}$, $7\hat{i}-8\hat{j}+9\hat{k}$, $3\hat{i}+20\hat{j}+5\hat{k}$ are coplanar.

Solution

$$\text{Let } 5\hat{i}+6\hat{j}+7\hat{k} = s(7\hat{i}-8\hat{j}+9\hat{k})+t(3\hat{i}+20\hat{j}+5\hat{k})$$

Equating the components, we have

$$7s+3t = 5$$
$$-8s+20t = 6$$
$$9s+5t = 7$$

Solving first two equations, we get, $s=t=\frac{1}{2}$, which satisfies the third equation.

Thus one vector is a linear combination of other two vectors.
Hence the given vectors are coplanar.

EXERCISE 8.2

(1) Verify whether the following ratios are direction cosines of some vector or not.

(i) $\frac{1}{5},\frac{3}{5},\frac{4}{5}$ (ii) $\frac{1}{\sqrt{2}},\frac{1}{2},\frac{1}{2}$ (iii) $\frac{4}{3},0,\frac{3}{4}$

(2) Find the direction cosines of a vector whose direction ratios are

(i) 1 , 2 , 3 (ii) 3 , – 1 , 3 (iii) 0 , 0 , 7

(3) Find the direction cosines and direction ratios for the following vectors.

(i) $3\hat{i}-4\hat{j}+8\hat{k}$ (ii) $3\hat{i}+\hat{j}+\hat{k}$ (iii) $\hat{j}$

(iv) $5\hat{i}-3\hat{j}-48\hat{k}$ (v) $3\hat{i}-3\hat{k}+4\hat{j}$ (vi) $\hat{i}-\hat{k}$

(4) A triangle is formed by joining the points (1, 0, 0), (0, 1, 0) and (0, 0, 1). Find the direction cosines of the medians.

(5) If $\frac{1}{2},\frac{1}{\sqrt{2}},a$ are the direction cosines of some vector, then find a.

(6) If (a , $a+b$, $a+b+c$) is one set of direction ratios of the line joining (1, 0, 0) and (0, 1, 0), then find a set of values of a, b, c.

(7) Show that the vectors

$2\hat{i}-\hat{j}+\hat{k},\ 3\hat{i}-4\hat{j}-4\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ form a right angled triangle.

(8) Find the value of λ for which the vectors $\vec{a}=3\hat{i}+2\hat{j}+9\hat{k}$ and $\vec{b}=\hat{i}+\lambda\hat{j}+3\hat{k}$ are parallel.

(9) Show that the following vectors are coplanar

(i) $\hat{i}-2\hat{j}+3\hat{k},\ -2\hat{i}+3\hat{j}-4\hat{k},\ -\hat{j}+2\hat{k}$

(ii) $2\hat{i}+3\hat{j}+\hat{k},\ \hat{i}-\hat{j},\ 7\hat{i}+3\hat{j}+2\hat{k}$.

(10) Show that the points whose position vectors $4\hat{i}+5\hat{j}+\hat{k},\ -\hat{j}-\hat{k},\ 3\hat{i}+9\hat{j}+4\hat{k}$ and $-4\hat{i}+4\hat{j}+4\hat{k}$ are coplanar.

(11) If $\vec{a}=2\hat{i}+3\hat{j}-4\hat{k},\ \vec{b}=3\hat{i}-4\hat{j}-5\hat{k}$, and $\vec{c}=-3\hat{i}+2\hat{j}+3\hat{k}$, find the magnitude and direction cosines of (i) $\vec{a}+\vec{b}+\vec{c}$ (ii) $3\vec{a}-2\vec{b}+5\vec{c}$.

(12) The position vectors of the vertices of a triangle are $\hat{i}+2\hat{j}+3\hat{k};\ 3\hat{i}-4\hat{j}+5\hat{k}$ and $-2\hat{i}+3\hat{j}-7\hat{k}$. Find the perimeter of the triangle.

(13) Find the unit vector parallel to

$3\vec{a}-2\vec{b}+4\vec{c}$ if $\vec{a}=3\hat{i}-\hat{j}-4\hat{k},\ \vec{b}=-2\hat{i}+4\hat{j}-3\hat{k}$, and $\vec{c}=\hat{i}+2\hat{j}-\hat{k}$.

(14) The position vectors $\vec{a},\vec{b},\vec{c}$ of three points satisfy the relation $2\vec{a}-7\vec{b}+5\vec{c}=\vec{0}$. Are these points collinear?

(15) The position vectors of the points P, Q, R, S are $i+j+k,\ 2\hat{i}+5\hat{j},\ 3\hat{i}+2\hat{j}-3\hat{k}$, and $\hat{i}-6\hat{j}-\hat{k}$ respectively. Prove that the line PQ and RS are parallel.

(16) Find the value or values of m for which $m\ (\hat{i}+\hat{j}+\hat{k})$ is a unit vector.

(17) Show that the points A (1, 1, 1), B(1, 2, 3) and C(2, – 1, 1) are vertices of an isosceles triangle.

8.8 Product of Vectors

We have seen the notion of addition of two vectors, subtraction of one vector from another vector and the multiplication of a vector by a scalar. Now we study the notion of product of two vectors. There are two ways of multiplying two vectors.

(i) scalar product (dot product) and

(ii) vector product (cross product).

To define such products we need the angle between two vectors.

8.8.1 Angle between two vectors

Let $\vec{a}$ and $\vec{b}$ be any two vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Angle between $\vec{a}$ and $\vec{b}$ is the angle between their directions when these directions are either both converge as in Fig. 8.36 or both diverge as in Fig. 8.34 from their point of intersection

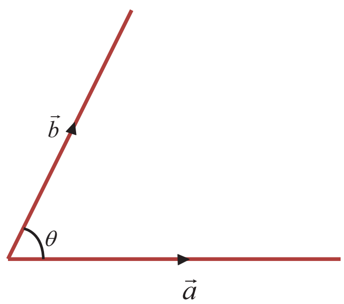


Fig. 8.34

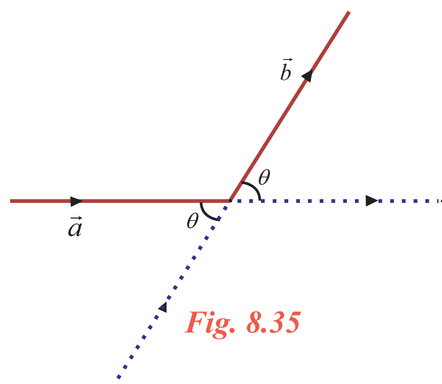


Fig. 8.35

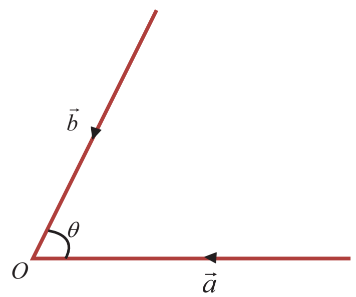


Fig. 8.36

Note that, if θ is the angle between two vectors then $0 \le \theta \le \pi$

When $\theta = 0$ or π, the vectors are parallel.

If two vectors neither converge nor diverge as in Fig. 8.35 then we can make them into either converge or diverge by extending the length of the vectors to find the angle between the two vectors.

8.8.2 Scalar product

Definition 8.16

Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors and θ be the included angle of the vectors as in Fig. 8.34.

Their **scalar product** or **dot product** is denoted by $\vec{a}\,.\,\vec{b}$ and is defined as a scalar $|\vec{a}||\vec{b}|\cos\theta$.

Thus $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Since the resultant of $\vec{a}\cdot\vec{b}$ is a scalar, it is called scalar product. Further we use the symbol dot ('$\cdot$') and hence another name dot product.

Geometrical meaning of scalar product (projection of one vector on another vector)

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and θ be the angle between $\vec{a}$ and $\vec{b}$.

Draw BL perpendicular to OA. From the right triangle OLB

$$\cos\theta = \frac{OL}{OB}$$

$$OL = OB\cos\theta = |\vec{b}|\cos\theta$$

But OL is the projection of $\vec{b}$ on $\vec{a}$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{a}|(OL)$$

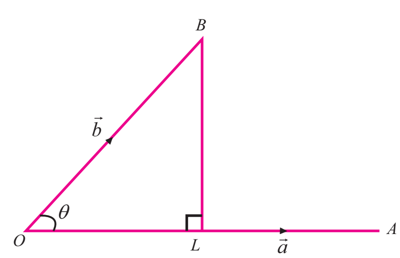


Fig. 8.37

$\vec{a}\cdot\vec{b} = |\vec{a}|$ (projection of $\vec{b}$ on $\vec{a}$)

Thus, projection of $\vec{b}$ on $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$.

In the same manner, projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{b}|}$.

8.8.3 Properties of Scalar Product

(i) Scalar product of two vectors is commutative.

With usual definition, $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta = |\vec{b}||\vec{a}|\cos\theta = \vec{b}\cdot\vec{a}$

That is, for any two vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

(ii) Nature of scalar product

We know that $0 \le \theta \le \pi$.

If $\theta = 0$ then $\vec{a}\cdot\vec{b} = ab$ [Two vectors are parallel in the same direction $\Rightarrow \theta = 0$].

If $\theta = \pi$ then $\vec{a}\cdot\vec{b} = -ab$ [Two vectors are parallel in the opposite direction $\Rightarrow \theta = \pi$.].

If $\theta = \dfrac{\pi}{2}$ then $\vec{a}\cdot\vec{b} = 0$ [Two vectors are perpendicular $\Rightarrow \theta = \dfrac{\pi}{2}$].

If $0 < \theta < \dfrac{\pi}{2}$ then $\cos\theta$ is positive and hence $\vec{a}\cdot\vec{b}$ is positive.

If $\dfrac{\pi}{2} < \theta < \pi$ then $\cos\theta$ is negative and hence $\vec{a}\cdot\vec{b}$ is negative.

That is, $\vec{a}\cdot\vec{b}$ is $\begin{cases} > 0 & \text{for } 0 \le \theta < \pi/2 \\ 0 & \text{for } \theta = \pi/2 \\ < 0 & \text{for } \pi/2 < \theta \le \pi \end{cases}$

(iii) $\vec{a}\cdot\vec{b} = 0 \Rightarrow |\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\theta = \dfrac{\pi}{2}$

(iv) For any two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0 \Leftrightarrow \vec{a}$ is perpendicular to $\vec{b}$.

(v) Different ways of representations of $\vec{a}\cdot\vec{a}$.

$\vec{a}\cdot\vec{a} = |\vec{a}|^2 = (\vec{a})^2 = \vec{a}^2 = a^2$.

These representations are essential while solving problems.

(vi) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$ (how?).

(vii) For any two scalars λ and μ

$\lambda\vec{a}\cdot\mu\vec{b} = \lambda\mu(\vec{a}\cdot\vec{b}) = (\lambda\mu\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\mu\vec{b})$.

(viii) Scalar product is distributive over vector addition.

That is, for any three vectors $\vec{a}, \vec{b}, \vec{c}$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ (Left distributivity)

$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$ (Right distributivity)

Subsequently,

$\vec{a}\cdot(\vec{b}-\vec{c}) = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}$ and $(\vec{a}-\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c}$

These can be extended to any number of vectors.

(ix) Vector identities

$$(\vec{a}+\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$$
$$(\vec{a}-\vec{b})^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}\cdot\vec{b}$$
$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$$

Proof

By property (iii)

$(\vec{a}+\vec{b})^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2 + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}\cdot\vec{b}$

Similarly one can prove other results.

(x) Working rule to find scalar product of two vectors

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\begin{aligned}\vec{a}\cdot\vec{b} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\\ &= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k})\\ &\quad + a_2b_1(\hat{j}\cdot\hat{i}) + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) + a_3b_3(\hat{k}\cdot\hat{k})\\ &= a_1b_1 + a_2b_2 + a_3b_3.\end{aligned}$$

Therefore, the scalar product of two vectors is equal to the sum of the products of their corresponding rectangular components.

(xi) If θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ then $\theta = \cos^{-1}\left[\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right]$.

(xii) For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$.

We know that if $\vec{a}$ and $\vec{b}$ are the two sides of a triangle then the sum $\vec{a}+\vec{b}$ represents the third side of the triangle. Therefore, by triangular property, $|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

(xiii) For any two vectors $\vec{a}$ and $\vec{b}$, $|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$.

If one of them is zero vector then the equality holds. So, let us assume that both are non-zero vectors. We have

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

That is, $\dfrac{|\vec{a}\cdot\vec{b}|}{|\vec{a}||\vec{b}|} = |\cos\theta| \le 1$

$$\Rightarrow |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|.$$

Example 8.11

Find $\vec{a}\cdot\vec{b}$ when

(i) $\vec{a} = \hat{i} - \hat{j} + 5\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{k}$

(ii) $\vec{a}$ and $\vec{b}$ represent the points (2, 3, – 1) and (– 1, 2, 3).

Solution

(i) $\vec{a}\cdot\vec{b} = (\hat{i} - \hat{j} + 5\hat{k})\cdot(3\hat{i} - 2\hat{k}) = (1)(3) + (-1)(0) + (5)(-2) = 3 - 10 = -7$

(ii) $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{a}\cdot\vec{b} = (2)(-1) + (3)(2) + (-1)(3) = -2 + 6 - 3 = 1.$

Example 8.12

Find $(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b})$ if $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = 3\hat{i}+2\hat{j}-\hat{k}$

Solution

$$(\vec{a}+3\vec{b})\cdot(2\vec{a}-\vec{b}) = 2\vec{a}\cdot\vec{a}+5\vec{a}\cdot\vec{b}-3\vec{b}\cdot\vec{b} = 2(1+1+4)+5(3+2-2)-3(9+4+1) = -15\,.$$

Example 8.13

If $\vec{a} = 2\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}$ be such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$ then find λ.

Solution

$$(\vec{a}+\lambda\vec{b})\cdot\vec{c} = 0 \Rightarrow \vec{a}\cdot\vec{c}+\lambda\vec{b}\cdot\vec{c} = 0$$
$$\Rightarrow (6+2)+\lambda(-3+2) = 0$$
$$\Rightarrow \lambda = 8.$$

Example 8.14

If $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ prove that $\vec{a}$ and $\vec{b}$ are perpendicular.

Solution

$$|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$$
$$|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$
$$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b} = |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}$$
$$\Rightarrow \quad 4\vec{a}\cdot\vec{b} = 0$$
$$\vec{a}\cdot\vec{b} = 0$$

Hence $\vec{a}$ and $\vec{b}$ are perpendicular.

Example 8.15

For any vector $\vec{r}$ prove that $\vec{r} = (\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

Solution

Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\vec{r}\cdot\hat{i} = \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{i} = x$$
$$\vec{r}\cdot\hat{j} = \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{j} = y$$
$$\vec{r}\cdot\hat{k} = \left(x\hat{i}+y\hat{j}+z\hat{k}\right)\cdot\hat{k} = z$$
$$(\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k} = x\hat{i}+y\hat{j}+z\hat{k} = \vec{r}$$

Thus $\vec{r} = (\vec{r}\cdot\hat{i})\hat{i}+(\vec{r}\cdot\hat{j})\hat{j}+(\vec{r}\cdot\hat{k})\hat{k}$.

Example 8.16

Find the angle between the vectors $5\hat{i}+3\hat{j}+4\hat{k}$ and $6\hat{i}-8\hat{j}-\hat{k}$.

Solution

Let $\vec{a} = 5\hat{i}+3\hat{j}+4\hat{k}$, and $\vec{b} = 6\hat{i}-8\hat{j}-\hat{k}$.
Let θ be the angle between them.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{30-24-4}{\sqrt{50}\ \sqrt{101}} = \frac{\sqrt{2}}{5\sqrt{101}} \quad \Rightarrow \theta = \cos^{-1}\left[\frac{\sqrt{2}}{5\sqrt{101}}\right].$$

Example 8.17

Find the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$ where *A, B, C, D* are the points (4, – 3, 0), (7, – 5, – 1), (– 2, 1, 3), (0, 2, 5).

Solution

Let *O* be the origin.

Therefore, $\overrightarrow{OA} = 4\hat{i} - 3\hat{j}$; $\overrightarrow{OB} = 7\hat{i} - 5\hat{j} - \hat{k}$; $\overrightarrow{OC} = -2\hat{i} + \hat{j} + 3\hat{k}$; $\overrightarrow{OD} = 2\hat{j} + 5\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\hat{i} - 2\hat{j} - \hat{k}$; $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = 2\hat{i} + \hat{j} + 2\hat{k}$

Projection of $\overrightarrow{AB}$ on $\overrightarrow{CD} = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{\left|\overrightarrow{CD}\right|} = \dfrac{6-2-2}{3} = \dfrac{2}{3}$.

Example 8.18

If $\vec{a}, \vec{b}$, and $\vec{c}$ are three unit vectors satisfying $\vec{a} - \sqrt{3}\,\vec{b} + \vec{c} = \vec{0}$ then find the angle between $\vec{a}$ and $\vec{c}$.

Solution

Let θ be the angle between $\vec{a}$ and $\vec{c}$.

$$\vec{a} - \sqrt{3}\,\vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \quad |(\vec{a}+\vec{c})| = |\sqrt{3}\,\vec{b}|$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{c}|\cos\theta = 3|\vec{b}|^2$$

$$\Rightarrow \quad 1 + 1 + (2)(1)(1)\cos\theta = 3(1)$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$$

Example 8.19

Show that the points (4, – 3, 1), (2, – 4, 5) and (1, – 1, 0) form a right angled triangle.

Solution

Trivially they form a triangle. It is enough to prove one angle is $\dfrac{\pi}{2}$. So find the sides of the triangle.

Let *O* be the point of reference and *A,B,C* be (4, – 3, 1), (2, – 4, 5) and (1, – 1, 0) respectively.

$\overrightarrow{OA} = 4\hat{i} - 3\hat{j} + \hat{k}$, $\overrightarrow{OB} = 2\hat{i} - 4\hat{j} + 5\hat{k}$, $\overrightarrow{OC} = \hat{i} - \hat{j}$

Now, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\hat{i} - \hat{j} + 4\hat{k}$

Similarly, $\overrightarrow{BC} = -\hat{i} + 3\hat{j} - 5\hat{k}$; $\overrightarrow{CA} = 3\hat{i} - 2\hat{j} + \hat{k}$

Clearly, $\overrightarrow{AB}\cdot\overrightarrow{CA} = 0$

Thus one angle is $\dfrac{\pi}{2}$. Hence they form a right angled triangle.

Note 8.5

Suppose three sides are given in vector form, prove

(i) either sum of the vectors is $\vec{0}$ or sum of any two vectors is equal to the third vector, to form a triangle.

(ii) dot product between any two vectors is 0 to ensure one angle is $\frac{\pi}{2}$.

EXERCISE 8.3

(1) Find $\vec{a}.\vec{b}$ when

(i) $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} - 4\hat{j} - 2\hat{k}$ (ii) $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$.

(2) Find the value λ for which the vectors $\vec{a}$ and $\vec{b}$ are perpendicular, where

(i) $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ (ii) $\vec{a} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \lambda\hat{k}$.

(3) If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}| = 10, |\vec{b}| = 15$ and $\vec{a}\cdot\vec{b} = 75\sqrt{2}$, find the angle between $\vec{a}$ and $\vec{b}$.

(4) Find the angle between the vectors

(i) $2\hat{i} + 3\hat{j} - 6\hat{k}$ and $6\hat{i} - 3\hat{j} + 2\hat{k}$ (ii) $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$.

(5) If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + 2\vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

(6) Show that the vectors $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$, and $\vec{c} = 3\hat{i} - 6\hat{j} + 2\hat{k}$ are mutually orthogonal.

(7) Show that the vectors $-\hat{i} - 2\hat{j} - 6\hat{k}, 2\hat{i} - \hat{j} + \hat{k}$, and $-\hat{i} + 3\hat{j} + 5\hat{k}$ form a right angled triangle.

(8) If $|\vec{a}| = 5$, $|\vec{b}| = 6$, $|\vec{c}| = 7$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.

(9) Show that the points (2, – 1, 3), (4, 3, 1) and (3, 1, 2) are collinear.

(10) If $\vec{a}, \vec{b}$ are unit vectors and θ is the angle between them, show that

(i) $\sin\frac{\theta}{2} = \frac{1}{2}|\vec{a} - \vec{b}|$ (ii) $\cos\frac{\theta}{2} = \frac{1}{2}|\vec{a} + \vec{b}|$ (iii) $\tan\frac{\theta}{2} = \frac{|\vec{a} - \vec{b}|}{|\vec{a} + \vec{b}|}$.

(11) Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}| = 3$, $|\vec{b}| = 4$, $|\vec{c}| = 5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$.

(12) Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$.

(13) Find λ, when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.

(14) Three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 3, |\vec{c}| = 4$, and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Find $4\vec{a}\cdot\vec{b} + 3\vec{b}\cdot\vec{c} + 3\vec{c}\cdot\vec{a}$.

8.8.4 Vector Product

To define vector product between two vectors, we need the concept of right handed and left handed system.

If we align the fingers of our right hand along the vector $\vec{a}$ and bend our fingers around in the direction of rotation from $\vec{a}$ towards $\vec{b}$ (through an angle less than $180°$), our thumb will point in the direction of $\vec{a}\times\vec{b}$. Now, following the right-hand rule, $\vec{b}\times\vec{a}$ will point in the direction opposite to $\vec{a}\times\vec{b}$ (See Fig. 8.38).

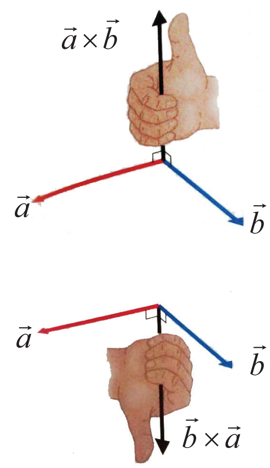
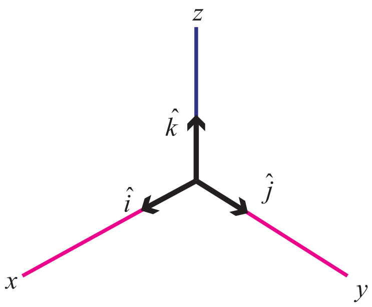


Fig. 8.38

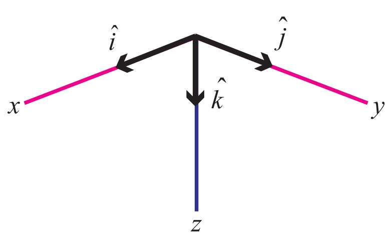
We may also say that if $\vec{a}$ is rotated into the direction of $\vec{b}$ through the angle $\theta(<\pi)$, then $\vec{a}\times\vec{b}$ advances in the same direction as a right-handed screw would if turned in the same way.

A Cartesian coordinate system is called right-handed if the corresponding unit vectors $\hat{i},\hat{j},\hat{k}$ in the positive direction of the axes form a right-handed triple as in Fig. 8.39. The system is called left handed if the sense of $\hat{k}$ is reversed as in Fig 8.40.



Right handed

Fig. 8.39



Left handed

Fig. 8.40

Right handed screw

Fi. 8.41

Left handed screw

Fig. 8.42

Definition 8.17

Vector product of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$ and is defined as

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

where θ is the angle between $\vec{a}$ and $\vec{b}, 0\le\theta\le\pi$.

Here $\vec{a},\vec{b},\hat{n}$ form a right handed system.

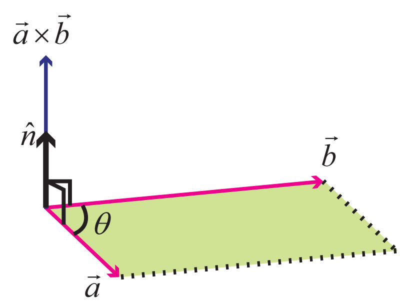


Fig. 8.43

The resultant is a vector with magnitude $|\vec{a}||\vec{b}|\sin\theta$ and has the direction $\hat{n}$.

Further $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$. That is, $\vec{a}\times\vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{b}$.

Note 8.6

(i) Note that the order of the vectors is very important to decide the direction of $\hat{n}$.

(ii) Since the resultant is a vector, this product is named as vector product. Again, we use the symbol cross '×' to define such a product and hence it has another name **cross product**.

Geometrical interpretation of vector product

Construct a parallelogram $OACB$ with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$ as adjacent sides.

Let $\lfloor AOB = \theta$

From the diagram,

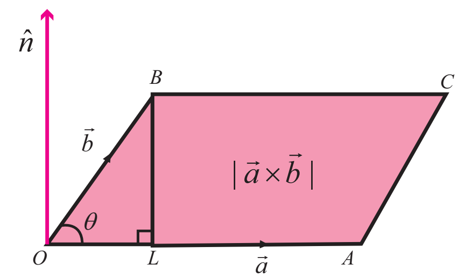


Fig. 8.44

$$\sin\theta = \frac{BL}{OB}$$

$$BL = (OB)\sin\theta = |\vec{b}|\sin\theta$$

Now $\quad |\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta = |\vec{a}|(BL)$

$= \text{(base) (height)} = \text{area of the parallelogram } OACB.$

Thus, $\vec{a}\times\vec{b}$ is a vector whose magnitude is the area of the parallelogram, having $\vec{a}$ and $\vec{b}$ as its adjacent sides and whose direction $\hat{n}$ is perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

Thus, $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.

From the area of the parallelogram, we can deduct the area of the triangle OAC as half of the area of $OACB$.

Deduction

The area of any triangle whose two sides are $\vec{a}$ and $\vec{b} = \frac{1}{2}|\vec{a}\times\vec{b}|$.

8.8.5 Properties

(i) Vector product is non-commutative

By definition

$$\vec{b}\times\vec{a} = |\vec{b}||\vec{a}|\sin\theta(-\hat{n})$$

[since $\vec{b}, \vec{a}, -\vec{n}$ form a right handed system]

$$= -|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

$$= -(\vec{a}\times\vec{b})$$

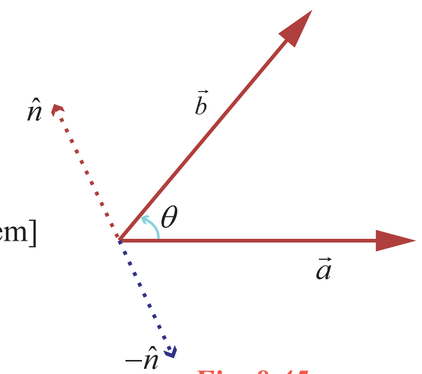


Fig. 8.45

Thus vector product is non-commutative.

(ii) If two vectors are collinear or parallel then $\vec{a}\times\vec{b} = \vec{0}$ (how?)

But $\vec{a}\times\vec{b} = \vec{0} \Rightarrow \quad \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ are parallel.

(iii) For any two non-zero vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\times\vec{b} = \vec{0} \Leftrightarrow \vec{a}$ and $\vec{b}$ are parallel.

Deduction

$$\vec{a}\times\vec{a} = \vec{0}$$

(iv) With usual meaning of $\hat{i}, \hat{j}$ and $\hat{k}$ (they form a right handed system), the following results are obtained.

It is clear that,

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$$

$$\hat{i}\times\hat{j} = \hat{k}\ ;\ \hat{j}\times\hat{k} = \hat{i}\ ;\ \hat{k}\times\hat{i} = \hat{j}$$

$$\hat{j}\times\hat{i} = -\hat{k}\ ;\ \hat{k}\times\hat{j} = -\hat{i}\ ;\ \hat{i}\times\hat{k} = -\hat{j} \quad \text{(how?)}$$

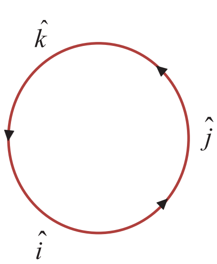


Fig. 8.46

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(v) If $\theta = \frac{\pi}{2}$ then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\hat{n}$.

(vi) For any scalars m and n,

$$m\vec{a}\times n\vec{b} = mn(\vec{a}\times\vec{b}) = (mn\vec{a})\times\vec{b} = \vec{a}\times(mn\vec{b}) = n\vec{a}\times m\vec{b} .$$

(vii) Vector product is distributive over addition.

That is,
$$\vec{a}\times(\vec{b}+\vec{c}) = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$$
$$(\vec{a}+\vec{b})\times\vec{c} = (\vec{a}\times\vec{c})+(\vec{b}\times\vec{c}).$$

This property can be extended to subtraction and to any number of vectors.

That is,
$$\vec{a}\times(\vec{b}-\vec{c}) = (\vec{a}\times\vec{b})-(\vec{a}\times\vec{c})$$
$$\vec{a}\times(\vec{b}+\vec{c}+\vec{d}) = (\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})+(\vec{a}\times\vec{d}).$$

(viii) Working rule to find the cross product

Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$.

$$\begin{aligned}
\vec{a}\times\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) \times (b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \\
&= a_1b_1(\hat{i}\times\hat{i})+a_1b_2(\hat{i}\times\hat{j})+a_1b_3(\hat{i}\times\hat{k})+a_2b_1(\hat{j}\times\hat{i}) \\
&\quad +a_2b_2(\hat{j}\times\hat{j})+a_2b_3(\hat{j}\times\hat{k})+a_3b_1(\hat{k}\times\hat{i})+a_3b_2(\hat{k}\times\hat{j})+a_3b_3(\hat{k}\times\hat{k}) \\
&= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}+(a_1b_2-a_2b_1)\hat{k}
\end{aligned}$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} .$$

(ix) If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\theta = \sin^{-1}\left[\frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}\right]$.

(The proof of this result is left as an exercise)

Note 8.7

In this case θ is always acute. Thus, if we try to find the angle using vector product, we get only the acute angle. Hence in problems of finding the angle, the use of dot product is preferable since it specifies the position of the angle θ.

(x) The unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm\hat{n} = \pm\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$ (how?)

Vectors of magnitude λ, perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm\lambda\hat{n} = \pm\lambda\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$.

Example 8.20

Find $|\vec{a}\times\vec{b}|$, where $\vec{a} = 3\hat{i}+4\hat{j}$ and $\vec{b} = \hat{i}+\hat{j}+\hat{k}$.

Solution

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(4-0)-\hat{j}(3-0)+\hat{k}(3-4) = 4\hat{i}-3\hat{j}-\hat{k} .$$

$$|\vec{a}\times\vec{b}| = |4\vec{i}-3\vec{j}-\vec{k}| = \sqrt{16+9+1} = \sqrt{26} .$$

Example 8.21

If $\vec{a} = -3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = 6\hat{i} + 2\hat{j} - 3\hat{k}$,

verify (i) $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

(ii) $\vec{b}$ and $\vec{a} \times \vec{b}$ are perpendicular to each other.

Solution :

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & -7 \\ 6 & 2 & -3 \end{vmatrix} = \hat{i}(-12+14) - \hat{j}(9+42) + \hat{k}(-6-24) = 2\hat{i} - 51\hat{j} - 30\hat{k}$$

(i) $\vec{a} \cdot (\vec{a} \times \vec{b}) = (-3\hat{i} + 4\hat{j} - 7\hat{k}) \cdot (2\hat{i} - 51\hat{j} - 30\hat{k}) = -6 - 204 + 210 = 0$.

Therefore, $\vec{a}$ and $\vec{a} \times \vec{b}$ are perpendicular.

(ii) $\vec{b} \cdot (\vec{a} \times \vec{b}) = (6\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (2\hat{i} - 51\hat{j} - 30\hat{k}) = 12 - 102 + 90 = 0.$

Therefore $\vec{b}$ and $\vec{a} \times \vec{b}$ are perpendicular.

Example 8.22

Find the vectors of magnitude 6 which are perpendicular to both vectors

$\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + j - 2\hat{k}$.

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = \hat{i}(2-3) - \hat{j}(-8+6) + \hat{k}(4-2) = -\hat{i} + 2\hat{j} + 2\hat{k}$$

Unit vectors perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm \dfrac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|} = \pm \left[\dfrac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3}\right]$

Vectors of magnitude 6 perpendicular to both $\vec{a}$ and $\vec{b}$ are $\pm 2(-\hat{i} + 2\hat{j} + 2\hat{k})$.

Example 8.23

Find the cosine and sine angle between the vectors $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 4\hat{i} - 2\hat{j} + 2\hat{k}$.

Solution

Let θ be the angle between $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = (2\hat{i} + \hat{j} + 3\hat{k}) \cdot (4\hat{i} - 2\hat{j} + 2\hat{k}) = 8 - 2 + 6 = 12$$

$$|\vec{a}| = |2\hat{i} + \hat{j} + 3\hat{k}| = \sqrt{14} \quad ; \quad |\vec{b}| = |4\hat{i} - 2\hat{j} + 2\hat{k}| = \sqrt{24}$$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{12}{\sqrt{14}\sqrt{24}} = \sqrt{\frac{3}{7}}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 4 & -2 & 2 \end{vmatrix} = \hat{i}(2+6) - \hat{j}(4-12) + \hat{k}(-4-4) = 8\hat{i} + 8\hat{j} - 8\hat{k}$$

$$|\vec{a} \times \vec{b}| = |8\hat{i} + 8\hat{j} - 8\hat{k}| = 8\sqrt{3}$$

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14}\sqrt{24}} = \frac{4\sqrt{3}}{\sqrt{7}\sqrt{12}} = \frac{2}{\sqrt{7}}.$$

Example 8.24

Find the area of the parallelogram whose adjacent sides are $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$.

Solution

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}(1+4) - \hat{j}(3-4) + \hat{k}(-3-1) = 5\hat{i} + \hat{j} - 4\hat{k} \; .$$

$$|\vec{a} \times \vec{b}| = |5\hat{i} + \hat{j} - 4\hat{k}| = \sqrt{42}$$

Area of the parallelogram is $\sqrt{42}$ sq.units.

Example 8.25

For any two vectors $\vec{a}$ and $\vec{b}$, prove that $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 \; |\vec{b}|^2$

Solution

We have, $\quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 \; |\vec{b}|^2 \sin^2\theta + |\vec{a}|^2 \; |\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2 \; |\vec{b}|^2 (\sin^2\theta + \cos^2\theta) = |\vec{a}|^2 \; |\vec{b}|^2 .$$

Example 8.26

Find the area of a triangle having the points $A(1,0,0), B(0,1,0),$ and $C(0,0,1)$ as its vertices.

Solution :

Let us find two sides of the triangle.

$$\overrightarrow{AB} = -\hat{i} + \hat{j} \; ; \; \overrightarrow{AC} = -\hat{i} + \hat{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \hat{i} + \hat{j} + \hat{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{3}$$

The area of the triangle ABC is $\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{\sqrt{3}}{2}$.

Note 8.8

Instead of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, one can take any two sides.

EXERCISE 8.4

(1) Find the magnitude of $\vec{a} \times \vec{b}$ if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.

(2) Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$.

(3) Find the vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane which contains $\hat{i} + 2\hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} + 4\hat{k}$.

(4) Find the unit vectors perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

(5) Find the area of the parallelogram whose two adjacent sides are determined by the vectors $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$.

(6) Find the area of the triangle whose vertices are $A(3,-1,2)$, $B(1,-1,-3)$ and $C(4,-3,1)$.

(7) If $\vec{a},\vec{b},\vec{c}$ are position vectors of the vertices A, B, C of a triangle ABC, show that the area of the triangle ABC is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$. Also deduce the condition for collinearity of the points $A, B,$ and C.

(8) For any vector $\vec{a}$ prove that $|\vec{a}\times\hat{i}|^2+|\vec{a}\times\hat{j}|^2+|\vec{a}\times\hat{k}|^2=2|\vec{a}|^2$.

(9) Let $\vec{a},\vec{b},\vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}=0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. Prove that $\vec{a}=\pm\frac{2}{\sqrt{3}}(\vec{b}\times\vec{c})$.

(10) Find the angle between the vectors $2\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$ using vector product.

BAZDFD

EXERCISE 8.5

Choose the correct or the most suitable answer from the given four alternatives

(1) The value of $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{DA}+\overrightarrow{CD}$ is

(1) $\overrightarrow{AD}$ (2) $\overrightarrow{CA}$ (3) $\vec{0}$ (4) $-\overrightarrow{AD}$

(2) If $\vec{a}+2\vec{b}$ and $3\vec{a}+m\vec{b}$ are parallel, then the value of m is

(1) 3 (2) $\frac{1}{3}$ (3) 6 (4) $\frac{1}{6}$

(3) The unit vector parallel to the resultant of the vectors $\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$ is

(1) $\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{5}}$ (2) $\frac{2\hat{i}+\hat{j}}{\sqrt{5}}$ (3) $\frac{2\hat{i}-\hat{j}+\hat{k}}{\sqrt{5}}$ (4) $\frac{2\hat{i}-\hat{j}}{\sqrt{5}}$

(4) A vector $\overrightarrow{OP}$ makes 60° and 45° with the positive direction of the x and y axes respectively. Then the angle between $\overrightarrow{OP}$ and the z-axis is

(1) 45° (2) 60° (3) 90° (4) 30°

(5) If $\overrightarrow{BA}=3\hat{i}+2\hat{j}+\hat{k}$ and the position vector of B is $\hat{i}+3\hat{j}-\hat{k}$, then the position vector A is

(1) $4\hat{i}+2\hat{j}+\hat{k}$ (2) $4\hat{i}+5\hat{j}$ (3) $4\hat{i}$ (4) $-4\hat{i}$

(6) A vector makes equal angle with the positive direction of the coordinate axes. Then each angle is equal to

(1) $\cos^{-1}\left(\frac{1}{3}\right)$ (2) $\cos^{-1}\left(\frac{2}{3}\right)$ (3) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (4) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$

(7) The vectors $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$ are

(1) parallel to each other (2) unit vectors

(3) mutually perpendicular vectors (4) coplanar vectors.

(8) If $ABCD$ is a parallelogram, then $\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}$ is equal to

(1) $2(\overrightarrow{AB}+\overrightarrow{AD})$ (2) $4\overrightarrow{AC}$ (3) $4\overrightarrow{BD}$ (4) $\vec{0}$

(9) One of the diagonals of parallelogram $ABCD$ with $\vec{a}$ and $\vec{b}$ as adjacent sides is $\vec{a}+\vec{b}$. The other diagonal $\overrightarrow{BD}$ is

(1) $\vec{a}-\vec{b}$ (2) $\vec{b}-\vec{a}$ (3) $\vec{a}+\vec{b}$ (4) $\dfrac{\vec{a}+\vec{b}}{2}$

(10) If $\vec{a},\vec{b}$ are the position vectors A and B, then which one of the following points whose position vector lies on AB, is

(1) $\vec{a}+\vec{b}$ (2) $\dfrac{2\vec{a}-\vec{b}}{2}$ (3) $\dfrac{2\vec{a}+\vec{b}}{3}$ (4) $\dfrac{\vec{a}-\vec{b}}{3}$

(11) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are the position vectors of three collinear points, then which of the following is true?

(1) $\vec{a}=\vec{b}+\vec{c}$ (2) $2\vec{a}=\vec{b}+\vec{c}$ (3) $\vec{b}=\vec{c}+\vec{a}$ (4) $4\vec{a}+\vec{b}+\vec{c}=\vec{0}$

(12) If $\vec{r}=\dfrac{9\vec{a}+7\vec{b}}{16}$, then the point P whose position vector $\vec{r}$ divides the line joining the points with position vectors $\vec{a}$ and $\vec{b}$ in the ratio

(1) 7 : 9 internally (2) 9 : 7 internally

(3) 9 : 7 externally (4) 7 : 9 externally

(13) If $\lambda\hat{i}+2\lambda\hat{j}+2\lambda\hat{k}$ is a unit vector, then the value of λ is

(1) $\dfrac{1}{3}$ (2) $\dfrac{1}{4}$ (3) $\dfrac{1}{9}$ (4) $\dfrac{1}{2}$

(14) Two vertices of a triangle have position vectors $3\hat{i}+4\hat{j}-4\hat{k}$ and $2\hat{i}+3\hat{j}+4\hat{k}$. If the position vector of the centroid is $\hat{i}+2\hat{j}+3\hat{k}$, then the position vector of the third vertex is

(1) $-2\hat{i}-\hat{j}+9\hat{k}$ (2) $-2\hat{i}-\hat{j}-6\hat{k}$ (3) $2\hat{i}-\hat{j}+6\hat{k}$ (4) $-2\hat{i}+\hat{j}+6\hat{k}$

(15) If $|\vec{a}+\vec{b}|=60, |\vec{a}-\vec{b}|=40$ and $|\vec{b}|=46$, then $|\vec{a}|$ is

(1) 42 (2) 12 (3) 22 (4) 32

(16) If $\vec{a}$ and $\vec{b}$ having same magnitude and angle between them is $60°$ and their scalar product is $\dfrac{1}{2}$ then $|\vec{a}|$ is

(1) 2 (2) 3 (3) 7 (4) 1

(17) The value of $\theta\in\left(0,\dfrac{\pi}{2}\right)$ for which the vectors $\vec{a}=(\sin\theta)\hat{i}+(\cos\theta)\hat{j}$ and $\vec{b}=\hat{i}-\sqrt{3}\hat{j}+2\hat{k}$ are perpendicular, is equal to

(1) $\dfrac{\pi}{3}$ (2) $\dfrac{\pi}{6}$ (3) $\dfrac{\pi}{4}$ (4) $\dfrac{\pi}{2}$

(18) If $|\vec{a}|=13, |\vec{b}|=5$ and $\vec{a}\cdot\vec{b}=60°$ then $|\vec{a}\times\vec{b}|$ is

(1) 15 (2) 35 (3) 45 (4) 25

(19) Vectors $\vec{a}$ and $\vec{b}$ are inclined at an angle $\theta=120°$. If $|\vec{a}|=1, |b|=2$, then $[(\vec{a}+3\vec{b})\times(3\vec{a}-\vec{b})]^2$ is equal to

(1) 225 (2) 275 (3) 325 (4) 300

(20) If $\vec{a}$ and $\vec{b}$ are two vectors of magnitude 2 and inclined at an angle $60°$, then the angle between $\vec{a}$ and $\vec{a}+\vec{b}$ is

(1) 30° (2) 60° (3) 45° (4) 90°

(21) If the projection of $5\hat{i}-\hat{j}-3\hat{k}$ on the vector $\hat{i}+3\hat{j}+\lambda\hat{k}$ is same as the projection of $\hat{i}+3\hat{j}+\lambda\hat{k}$ on $5\hat{i}-\hat{j}-3\hat{k}$, then λ is equal to

(1) ± 4 (2) ± 3 (3) ± 5 (4) ± 1

(22) If (1, 2, 4) and (2, $-3\lambda-3$) are the initial and terminal points of the vector $\hat{i}+5\hat{j}-7\hat{k}$, then the value of λ is equal to

(1) $\frac{7}{3}$ (2) $-\frac{7}{3}$ (3) $-\frac{5}{3}$ (4) $\frac{5}{3}$

(23) If the points whose position vectors $10\hat{i}+3\hat{j}$, $12\hat{i}-5\hat{j}$ and $a\hat{i}+11\hat{j}$ are collinear then a is equal to

(1) 6 (2) 3 (3) 5 (4) 8

(24) If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+x\hat{j}+\hat{k}$, $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$ and $\vec{a}\cdot(\vec{b}\times\vec{c})=70$, then x is equal to

(1) 5 (2) 7 (3) 26 (4) 10

(25) If $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}, |\vec{b}|=5$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then the area of the triangle formed by these two vectors as two sides, is

(1) $\frac{7}{4}$ (2) $\frac{15}{4}$ (3) $\frac{3}{4}$ (4) $\frac{17}{4}$

SUMMARY

In this chapter we have acquired the knowledge of the following :

- A **scalar** is a quantity that is determined by its magnitude.
- A **vector** is a quantity that is determined by both its magnitude and its direction
- If we have a liberty to choose the origins of the vector at any point then it is said to be a **free vector,** whereas if it is restricted to a certain specified point then the vector is said to be a **localized vector.**
- Two or more vectors are said to be **coplanar** if they lie on the same plane or parallel to the same plane.
- Two vectors are said to be **equal** if they have equal length and the same direction.
- A vector of magnitude 0 is called the **zero vector.**
- A vector of magnitude 1 is called a **unit vector**.

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- Let $\vec{a}$ be a vector and m be a scalar. Then the vector $m\vec{a}$ is called the **scalar multiple of a vector** $\vec{a}$ by the scalar m.
- Two vectors $\vec{a}$ and $\vec{b}$ are said to be **parallel** if $\vec{a} = \lambda\vec{b}$, where λ is a scalar.
- If $\vec{a}, \vec{b}$ and $\vec{c}$ are the sides of a triangle taken in order then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$
- Vector addition is associative.
- For any vector $\vec{a}$, $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$.
- For any vector $\vec{a}$, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$.
- Vector addition is commutative.
- "If two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order". This is known as the **triangle law of addition.**
- In a parallelogram $OABC$, if $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represents two adjacent sides, then the diagonal $\overrightarrow{OC}$ represents their sum. This is **parallelogram law of addition**.
- If α, β, γ are the direction angles then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines.
- The direction ratios of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are x, y, z.
- If $\vec{a}, \vec{b}$ and $\vec{c}$ are three non-coplanar vectors in the space, then any vector in the space can be written as $l\vec{a} + m\vec{b} + n\vec{c}$ in a unique way.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point and let α, β, γ be the direction angles of $\vec{r}$. Then

(i) the sum of the squares of the direction cosines of $\vec{r}$ is 1.

(ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

(iii) the direction cosines of $\vec{r}$ are $\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}$.

(iv) l, m, n are the direction cosines of a vector if and only if $l^2 + m^2 + n^2 = 1$.

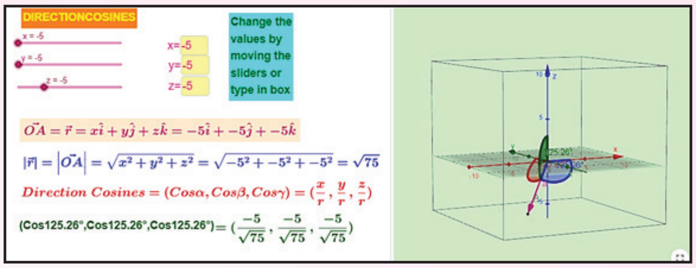
(v) any unit vector can be written as $\cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$.

- The **scalar product** of the vectors $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$.
- **Vector product** of any two non-zero vectors $\vec{a}$ and $\vec{b}$ is written as $\vec{a}\times\vec{b}$ and is defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}, 0 \le \theta \le \pi$. Here $\vec{a}, \vec{b}, \hat{n}$ form a right handed system.

ICT CORNER 8(a)

Vector Algebra

Expected Outcome



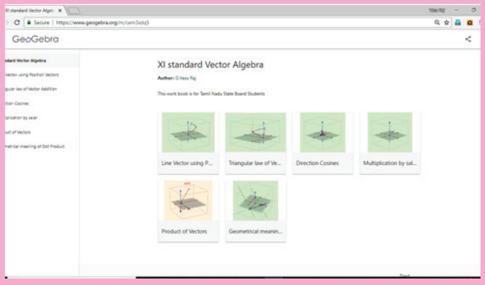
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI Standard Vector Algebra" will appear. In that there are several worksheets related to your lesson.

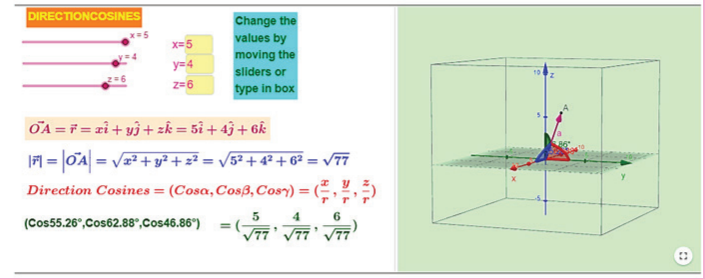
Step 2

Select the work sheet "Direction Cosines". 3-D representation is found on Right side. You can rotate 3-D picture by right clicking on the mouse to see various positions

You can move the sliders or entering x, y and z values to change the vector.



Step1



Step2

Browse in the link:

XI Standard Vector Algebra: https://ggbm.at/cem3sdq5

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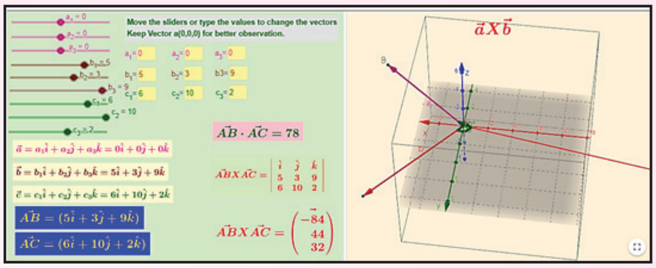
ICT CORNER 8(b)

Vector Algebra

Expected Outcome



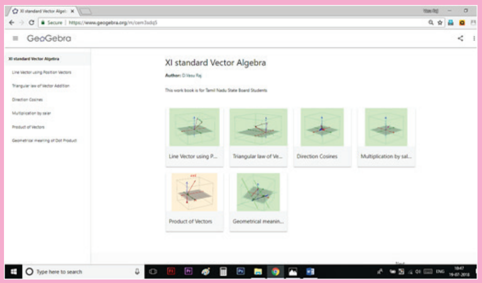
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI Standard Vector Algebra" will appear. In that there are several worksheets related to your lesson.

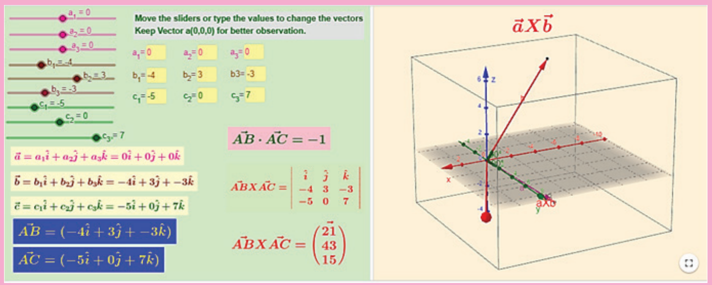
Step 2

Select the work sheet "Product of Vectors". 3-D representation is found on Right side. You can rotate 3-D picture by right clicking on the mouse to see various positions

You can move the sliders or entering x, y and z values to change the vector.(for clear understanding do not change a_1,a_2, and a_3 values. For ABXAC, components are given vertically)



Step1



Step2

Browse in the link:

XI Standard Vector Algebra: https://ggbm.at/cem3sdq5

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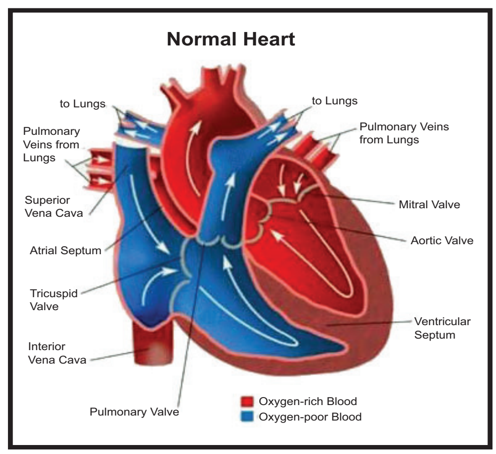
Chapter 9 Differential Calculus - Limits and Continuity

"Men pass away, but their deeds abide"

- Augustin-Louis Cauchy

9.1 Introduction

Calculus is about rates of change. Rates of change occur in all the sciences. A **mathematician** is interested in measuring the rate of change of the deviations of a straight line at a point on a curve, while a **physicist** is interested in the rate of change of displacement, and the velocity of a moving object. A **chemist** wants to know the rate of a chemical reaction that would result in the formation of one or more substances (called products) from one or more starting materials (called reactants).

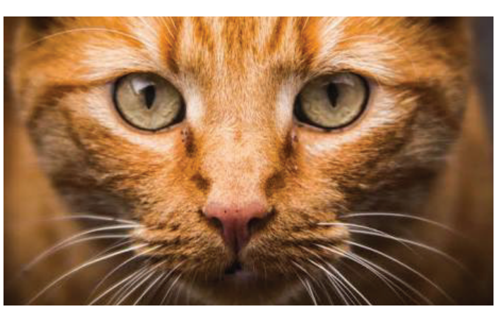


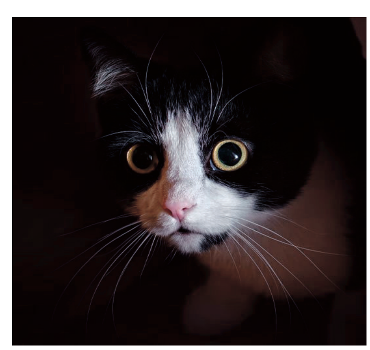
A **biologist** would like to analyse the changes that take place in the number of individuals in an animal population or plant population at any time; he would also want to know the rate at which blood flows through a blood vessel, such as a vein or artery and the part of the vessel / artery in which this flow is lowest or highest.

An **economists** also studies marginal demand, marginal revenue, and marginal profit, which are drawn from rates of change (that is, derivatives) of this demand, revenue and profit functions.

A **geologist** is interested in knowing the rate at which an intruded body of molten rock cools by conduction of heat into surrounding rocks. An **engineer** wants to know the rate at which water flows into or out of a reservoir. An urban **geographer** is interested in the rate of change of population density in a city with the expansion of the city. A **meteorologist** is concerned with the rate of change of atmospheric pressure with respect to height.

In **psychology**, those interested in learning theory, study the so called learning curve, which graphs the performance of someone learning a skill as a function of the training time. Of particular interest is the rate at which performance improves as time passes.





When we enter a darkened room, our eyes adjust to the reduced level of light by increasing the size of our pupils, allowing more light to enter the eyes and making objects around us easier to see. By contrast, when we enter a brightly lit room, our pupils contract, reducing the amount of light entering the eyes, as too much light would overload our visual system. Researchers study such mechanisms based on limits.

Velocity, density, current, power and temperature gradient in physics; rate of reaction and compressibility in chemistry, rate of growth and blood velocity in biology; marginal cost and marginal profit in economics; rate of heat flow in geology; rate of improvement of performance in psychology – these are all cases of a single mathematical concept, the derivative.

This is an illustration of the fact that part of the power of mathematics lies in its abstractness. A single abstract mathematical concept (such as derivatives) can have different interpretations in each of the sciences. When we develop the properties of the mathematical concept, we can then apply these results to all of the sciences. This is much more efficient than developing properties of special concepts in each separate science.

One of the greatest creations of the ancient past was Euclidean geometry. This monumental work was not matched in importance until the discovery of calculus almost two thousand years later.

Calculus was created independently in England by **Sir Isaac Newton** (1642 – 1727) and in Germany by **Gottfried Wilhelm Leibnitz** (1646 – 1716) in the last quarter of the seventeenth century.

Newton's interest in mathematics began with his study of two of the great books on mathematics at that time: **Euclid's Elements and Descartio La Geometric**. He also became aware of the work of the great scientists who preceded him, including Galileo and Fermat.

By the end of 1664, Newton seemed to have mastered all the mathematical knowledge of the time and had begun adding substantially to it. In 1665, he began his study of the rates of change or flexions, of quantities, such as distances or temperatures that varied continuously. The result of this study was what we today call differential calculus. All who study mathematics today stand on Isaac Newton's shoulders.

Many of Leibnitz's mathematical papers appeared in the journal 'Acta Eruditorum' which he cofounded in 1682. This journal contained his work on calculus and led to the bitter controversy with Newton over who first discovered calculus. Leibnitz was the first to publish the important results on calculus and was the first to use the notation that has now become standard.

Augustin – Louis Cauchy
(1789 – 1857)

Augustin-Louis Cauchy (1789 – 1857), born in Paris in 1789 is considered to be the most outstanding mathematical analyst of the first half of the nineteenth century. Cauchy made many contributions to calculus. In his 1829 text book '**Lecons le calcul differential**', he gave the first reasonably clear definition of a limit and was the first to define the derivative as the limit of the difference quotient,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Karl Weierstrass (1815-1897), a German mathematician gave the precise definition $(\in - \delta$ definition$)$ of the concepts of limit, continuity and differentiability.

What is Calculus?

Calculus is the mathematics of ratio of change of quantities. It is also the mathematics of tangent lines, slopes, areas, volumes, arc lengths, centroids, curvatures and a variety of other concepts that have enabled scientists, engineers and economists to model real-life situations.

Although pre calculus mathematics deals with velocities, accelerations, tangent lines, slopes and so on, there is a fundamental difference between pre calculus mathematics and calculus. Pre calculus mathematics is more static, whereas calculus is more dynamic. Here are some examples:

★ An object travelling at a constant velocity can be analysed with pre calculus mathematics. To analyse the velocity of an accelerating object, you need calculus.

★ The slope of a line can be analysed with pre calculus mathematics. To analyse the slope of a curve, you need calculus.

★ A tangent line to a circle can be analysed with pre calculus mathematics. To analyse a line tangential to a general graph, you need calculus.

★ The area of a rectangle can be analysed with pre calculus mathematics. To analyse the area under a general curve, you need calculus.

Each of these situations involves the same general strategy, the reformulation of pre calculus mathematics through the use of a limit process. So, one way to answer the question 'What is calculus?' is to say that calculus is a 'limit machine' that involves three stages. The first stage is pre calculus mathematics such as the slope of a line or the area of a rectangle. The second stage is the limit process and the third stage is a new calculus formulation, such as a derivative or integral.

Pre calculus Mathematics	$\Rightarrow$	Limit Process	$\Rightarrow$	Calculus

It is cautioned that those who try to learn calculus as if it were simply a collection of new formulae rather than as a process, will miss a great deal of understanding, self-confidence and satisfaction.

Learning Objectives

On completion of this chapter, the students are expected to

- visualize the concept of limit / continuity through geometric process.
- relate the concept of limit / continuity with every day life activities.
- assimilate limit / continuity as the heart and spirit of calculus.
- understand limit / continuity as an operation (operator) to measure / quantify / mathmatize changes in physical world.
- concretize the concept of limit / continuity via illustrations of real life related situations.

9.2 Limits

9.2.1 The calculation of limits

The notion of a limit, which we will discuss extensively in this chapter, plays a central role in calculus and in much of modern mathematics. However, although mathematics dates back over three thousand years, limits were not really understood until the monumental work of the great French mathematician Augustin – Louis Cauchy and Karl Weierstrass in the nineteenth century, the age of rigour in mathematics.

In this section we define limit and show how limits can be calculated.

Illustration 9.1

We begin by looking at the function $y = f(x) = x^2 + 3$. Note that f is a function from $\mathbb{R} \rightarrow \mathbb{R}$.

Let us investigate the behaviour of this function near $x = 2$. We can use two sets of x values : one set that approaches 2 from the left (values less than 2) and one set that approaches 2 from the right (values greater than 2) as shown in the table.

	x approaches 2 from the left							x approaches 2 from the right				
x	1.7	1.9	1.95	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.05	2.1	2.3
$f(x)$	5.89	6.61	6.8025	6.9601	6.99601	6.99960001	7	7.0040001	7.004001	7.0401	7.2025	7.41	8.29

It appears from the table that as x gets close to $x=2, f(x)=x^2+3$ gets close to 7. This is not surprising since if we now calculate $f(x)$ at $x = 2$, we obtain $f(2) = 2^2 + 3 = 7$.

In order to guess at this limit, we didn't have to evaluate $x^2 + 3$ at $x = 2$.

That is, as x approaches 2 from either the left (values lower than 2) or right (values higher than 2) the functional values $f(x)$ are approaching 7 from either side; that is, when x is near 2, $f(x)$ is near 7. The above situation is described in a condensed form:

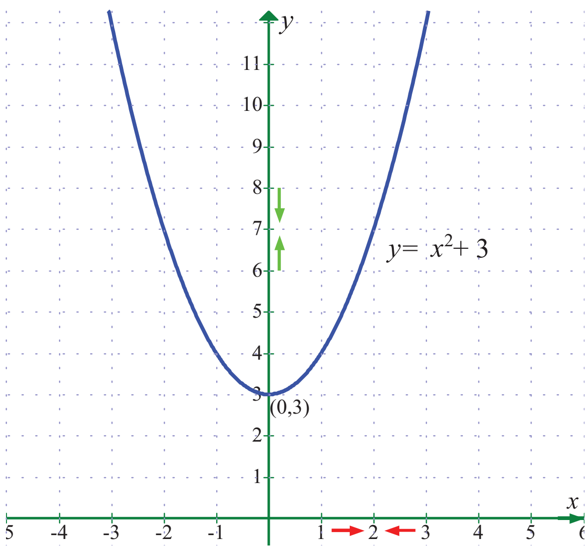


Fig. 9.1

The value 7 is the left limit of $f(x)$ as x approaches 2 from the left as well as 7 is the right limit of $f(x)$ as x approaches 2 from the right and write :

$f(x) \to 7$ as $x \to 2^-$ and $f(x) \to 7$ as $x \to 2^+$

or

$$\lim_{x\to 2^-} f(x) = 7 \text{ and } \lim_{x\to 2^+} f(x) = 7.$$

Note also that $\lim\limits_{x\to 2^-} f(x) = 7 = \lim\limits_{x\to 2^+} f(x)$. The common value is written as $\lim\limits_{x\to 2} f(x) = 7$.

We also observe that the limit is a definite real number. Here, definiteness means that $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$ are the same and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \text{ is a unique real number.}$$

The figure in Fig. 9.1 explains the geometrical significance of the above discussion of the behaviour of $f(x) = x^2 + 3$ as $x \to 2$.

Illustration 9.2

Next, let us look at the rational function $f(x) = \dfrac{16-x^2}{4+x}$.

The domain of this function is $\mathbb{R} \setminus \{-4\}$. Although $f(-4)$ is not defined, nonetheless, $f(x)$ can be calculated for any value of x near -4 because the symbol $\lim\limits_{x\to -4}\left(\dfrac{16-x^2}{4+x}\right)$ says that we consider values of x that are close to -4 but not equal to -4. The table below gives the values of $f(x)$ for values of x that approach -4.

$(x < -4)$ $(x \to -4^-)$	$f(x)$	$(x > -4)$ $(x \to -4^+)$	$f(x)$
-4.1	8.1	-3.9	7.9
-4.01	8.01	-3.99	7.99
-4.001	8.001	-3.999	7.999

For $x \neq -4$, $f(x)$ can be simplified by cancellation :

$$f(x) = \frac{16-x^2}{4+x}$$

$$= \frac{(4+x)(4-x)}{(4+x)} = 4 - x.$$

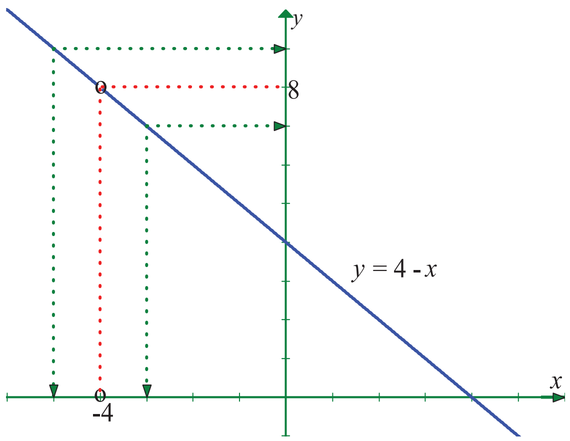


Fig. 9.2

As seen in Fig.9.2, the graph of $f(x)$ is essentially the graph of $y = 4 - x$ with the exception that the graph of f has a hole (puncture) at the point that corresponds to $x = -4$. As x gets closer and closer to -4, represented by the two arrow heads on the x-axis, the two arrow heads on the y-axis simultaneously get closer and closer to the number 8.

Here, note that

$$\lim_{x\to -4^-} f(x) = 8 = \lim_{x\to -4^+} f(x) \text{ and hence } \lim_{x\to -4} f(x) = \lim_{x\to -4} \frac{16-x^2}{4+x} = 8.$$

In Illustration 9.2, note that the function is not defined at $x = -4$ and yet $f(x)$ appears to be approaching a limit as x approaches -4. This often happens, and **it is important to realise that the existence or non-existence of $f(x)$ at $x = -4$ has no bearing on the existence of the limit of $f(x)$ as x approaches -4.**

Illustration 9.3

Now let us consider a function different from Illustrations 9.1 and 9.2.

Let $f(x) = \frac{|x|}{x}$.

$x = 0$ does not belong to the domain of this function, $\mathbb{R} \setminus \{0\}$. Look at the graph of this function. From the graph one can see that for positive values of x,

$$\frac{|x|}{x} = \frac{x}{x} = +1 \text{ and}$$

for negative x values, $\frac{|x|}{x} = \frac{-x}{x} = -1$.

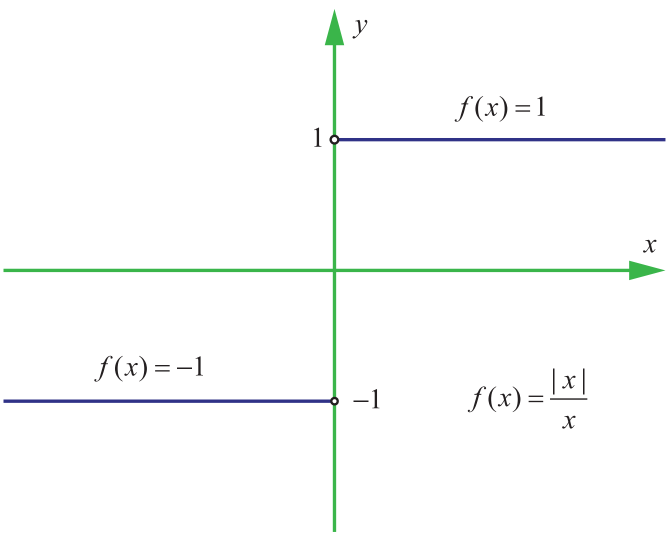


Fig. 9.3

This means that no matter how close x gets to 0 (in a small neighbourhood of 0), there will be both positive and negative x values that yield $f(x) = 1$ and $f(x) = -1$.

That is, $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = +1$.

This means that the limit does not exist. Of course, for any other value of x, there is a limit.

For example $\lim_{x\to 2^-} \frac{|x|}{x} = 1$ and $\lim_{x\to 2^+} \frac{|x|}{x} = 1$.

$$\text{Similarly,} \quad \lim_{x\to -3^-} \frac{|x|}{x} = \lim_{x\to -3^-} \frac{-x}{x} = -1$$

$$\lim_{x\to -3^+} \frac{|x|}{x} = \lim_{x\to -3^+} \frac{-x}{x} = -1.$$

In fact, for any real number $x_0 \neq 0$, $\lim\limits_{x\to x_0^-} \frac{|x|}{x} = -1 = \lim\limits_{x\to x_0^+} \frac{|x|}{x}$ if $x_0 < 0$ and

$$\lim_{x\to x_0^-} \frac{|x|}{x} = 1 = \lim_{x\to x_0^+} \frac{|x|}{x} \text{ if } x_0 > 0.$$

We call the attention of the reader to observe the differences reflected in Illustrations 9.1 to 9.3. In Illustration 9.1, the function $f(x) = x^2 + 3$ is defined at $x = 2$. i.e., 2 belongs to the domain of f namely $\mathbb{R} = (-\infty, \infty)$. In Illustration 9.2, the function is not defined at $x = -4$. In the former case we say the limit, $\lim\limits_{x\to 2} f(x)$ exists as x gets closer and closer to 2 to mean that $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$ stand for a unique real number. In the later case, although it is not defined at $x = -4$, $\lim\limits_{x\to -4} f(x)$ exist as x gets closer and closer to -4. In Illustration 9.3, $\lim\limits_{x\to 0} \frac{|x|}{x}$ does not exist to mean that the one sided limits $\lim\limits_{x\to 0^-} \frac{|x|}{x}$ and $\lim\limits_{x\to 0^+} \frac{|x|}{x}$ are different as x gets sufficiently close to 0. In the light of these observations we have the **intuitive** notion of limit as in

Definition 9.1

Let I be an open interval containing $x_0 \in \mathbb{R}$. Let $f : I \to \mathbb{R}$. Then we say that the limit of $f(x)$ is L, as x approaches x_0 [symbollically written as $\lim\limits_{x\to x_0} f(x) = L$], if, whenever x becomes sufficiently close to x_0 from either side with $x \neq x_0$, $f(x)$ gets sufficiently close to L.

The following (Fig 9.4 and 9.5) graphs depict the above narrations.

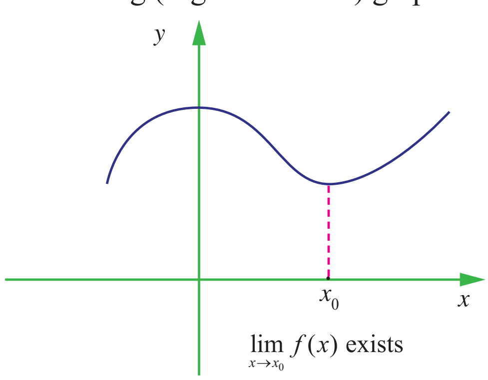


Fig. 9.4

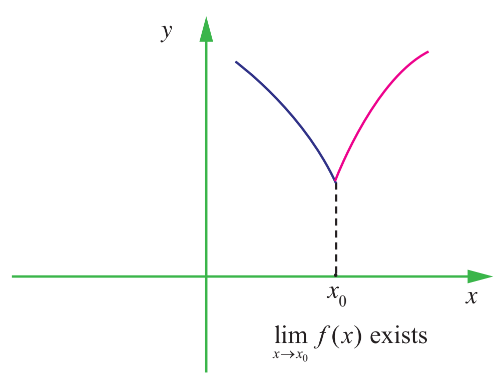


Fig. 9.5

9.2.2 One sided limits

Definition 9.2

We say that the **left-hand limit** of $f(x)$ as x approaches x_0 (or the limit of $f(x)$ as x approaches x_0 from the left) is equal to l_1 if we can make the values of $f(x)$ arbitrarily close to l_1 by taking x to be sufficiently close to x_0 and less than x_0. It is symbolically written as $f(x_0^-) = \lim\limits_{x\to x_0^-} f(x) = l_1$.

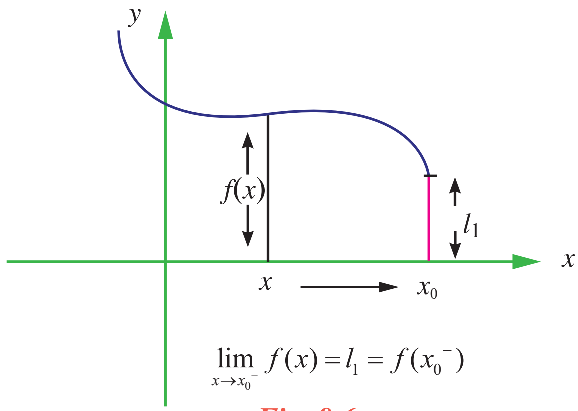
Similarly, we define the right hand limit.

Definition 9.3

We say that the **right-hand limit** of $f(x)$ as x approaches x_0 (or the limit of $f(x)$ as x approaches x_0 from the right) is equal to l_2 if we can make the values of $f(x)$ arbitrarily close to l_2 by taking x to be sufficiently close to x_0 and greater than x_0. It is symbolically written as $f(x_0^+) = \lim\limits_{x \to x_0^+} f(x) = l_2$.

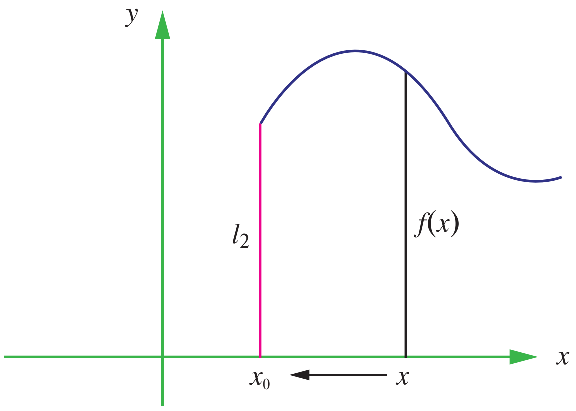
Thus the symbols "$x \to x_0^-$" and "$x \to x_0^+$" mean that we consider only $x < x_0$ and $x > x_0$ respectively.

These definitions are illustrated in the following Fig. 9.6 to 9.9.



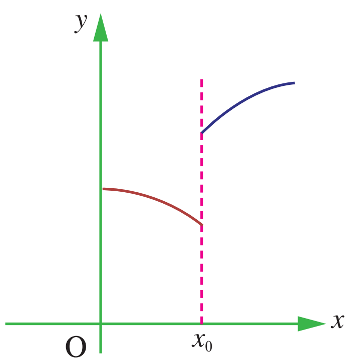
$\lim\limits_{x \to x_0^-} f(x) = l_1 = f(x_0^-)$

Fig. 9.6



$\lim\limits_{x \to x_0^+} f(x) = l_2 = f(x_0^+)$

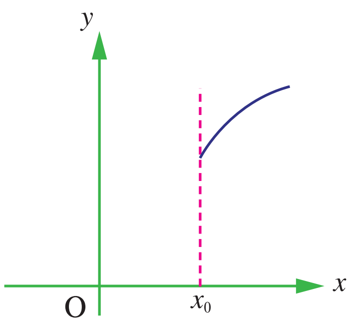
Fig. 9.7



$\lim\limits_{x \to x_0} f(x)$ does not exist

Fig. 9.8

(Different values are obtained as x_0 is approached from the left and from the right)



$\lim\limits_{x \to x_0} f(x)$ does not exist

Fig. 9.9

(Function not defined to the left of x_0)

From the above discussions we conclude that $\lim\limits_{x \to x_0} f(x) = L$ exists if the following hold :

(i) $\lim\limits_{x \to x_0^+} f(x)$ exists,

(ii) $\lim\limits_{x \to x_0^-} f(x)$ exists and

(iii) $\lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0^-} f(x) = L$.

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From the definitions of one sided limits and that of the limit of $f(x)$ as we have the following :

$\lim\limits_{x \to x_0} f(x) = L$ iff $\lim\limits_{x \to x_0^-} f(x) = L = \lim\limits_{x \to x_0^+} f(x)$.

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Thus, when we say $\lim\limits_{x \to x_0} f(x)$ exists, it is understood that L is a unique real number. If any one of the above conditions fails then we say, the limit of $f(x)$ as x approaches x_0 does not exist.

We remark that the existence of one sided limits is weaker than the existence of limits.

Sometimes it is very useful to use the following in computing left and right limits. For $h > 0$,

$f(x_0^-) = \lim\limits_{h \to 0} f(x_0 - h)$ and $f(x_0^+) = \lim\limits_{h \to 0} f(x_0 + h)$.

Note that $f(x_0^-)$ and $f(x_0^+)$ stand for the left and right limiting values. But $f(x_0)$ is the value of the function at $x = x_0$.

Example 9.1

Calculate $\lim\limits_{x \to 0} |x|$.

Solution

Recall from the earlier chapter 1

that $|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$

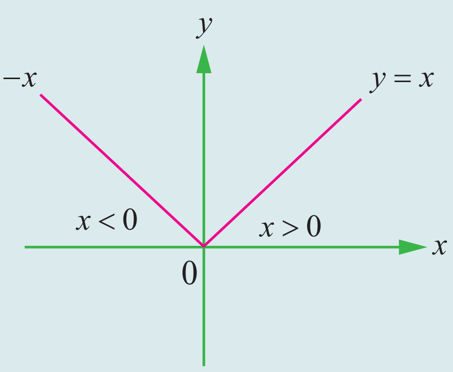


Fig. 9.10

If $x > 0$, then $|x| = x$, which tends to 0 as

$x \to 0$ from the right of 0. That is, $\lim\limits_{x \to 0^+} |x| = 0$

If $x < 0$, then $|x| = -x$ which again tends to 0 as $x \to 0$ from the left of 0. That is, $\lim\limits_{x \to 0^-} |x| = 0$.

Thus, $\lim\limits_{x \to 0^-} |x| = 0 = \lim\limits_{x \to 0^+} |x|$.

Hence $\lim\limits_{x \to 0} |x| = 0$.

Example 9.2

Consider the function $f(x) = \sqrt{x}$, $x \geq 0$.

Does $\lim\limits_{x \to 0} f(x)$ exist?

Solution

No. $f(x) = \sqrt{x}$ is not even defined for $x < 0$.

Therefore as $x \to 0^-$, $\lim\limits_{x \to 0^-} \sqrt{x}$ does not exist.

However, $\lim\limits_{x \to 0^+} \sqrt{x} = 0$. Therefore $\lim\limits_{x \to 0} \sqrt{x}$ does not exist.

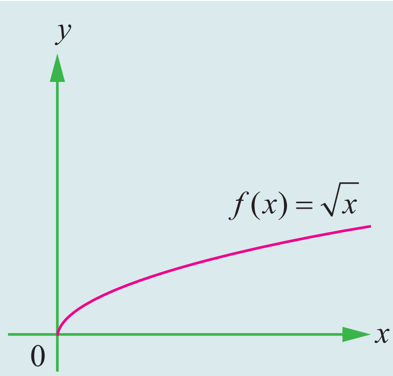


Fig. 9.11

Does $\lim\limits_{x \to 0^-} \log x$ exist?

Look at the graph of $\log x$ for the answer.

Example 9.3

Evaluate $\lim_{x\to 2^-}\lfloor x\rfloor$ and $\lim_{x\to 2^+}\lfloor x\rfloor$.

Solution

The greatest integer function $f(x)=\lfloor x\rfloor$ is defined as the greatest integer lesser than or equal to x.

From the graph (Fig. 1.25) of this function it is clear that $\lim_{x\to 2^-}\lfloor x\rfloor=1$ and $\lim_{x\to 2^+}\lfloor x\rfloor=2$.

Moreover, for any integer n, $\lim_{x\to n^-}\lfloor x\rfloor=n-1$ and $\lim_{x\to n^+}\lfloor x\rfloor=n$.

Does $f(x)=\lim_{x\to n}\lceil x\rceil$ exist? Look at the graph of $\lceil x\rceil$ (Fig.1.26) for the answer.

Example 9.4

Let $f(x)=\begin{cases} x+1, & x>0 \\ x-1, & x<0 \end{cases}$.

Verify the existence of limit as $x\to 0$.

Solution

The function is graphed in Fig.9.12.

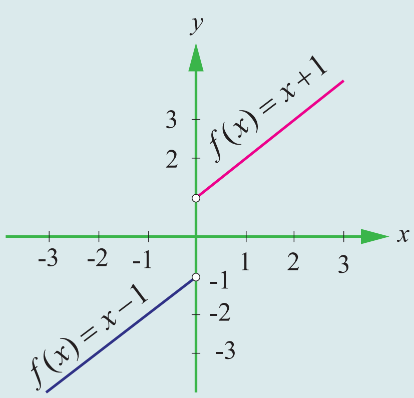


Fig. 9.12

Clearly $\lim_{x\to 0^-}f(x)=-1$ and $\lim_{x\to 0^+}f(x)=1$. Since these limits are different, $\lim_{x\to 0}f(x)$ does not exist.

Example 9.5

Check if $\lim_{x\to -5}f(x)$ exists or not, where $f(x)=\begin{cases} \dfrac{|x+5|}{x+5}, & \text{for } x\neq -5 \\ 0, & \text{for } x=-5 \end{cases}$

Solution

(i) $f(-5^-)$.

For $x<-5$, $|x+5|=-(x+5)$

Thus $f(-5^-)=\lim_{x\to -5^-}\dfrac{-(x+5)}{(x+5)}=-1$

(ii) $f(-5^+)$.

For $x>-5$, $|x+5|=(x+5)$

Thus $f(-5^+)=\lim_{x\to -5^+}\dfrac{(x+5)}{(x+5)}=1$

Note that $f(-5^-)\neq f(-5^+)$. Hence the limit does not exist.

Example 9.6

Test the existence of the limit, $\lim_{x\to 1} \frac{4|x-1|+x-1}{|x-1|}$, $x \neq 1$.

Solution

For $x > 1$, $|x-1| = x-1$ and $f(1^+) = \lim_{x\to 1^+} \frac{4(x-1)+x-1}{x-1} = \lim_{x\to 1^+} \frac{5(x-1)}{(x-1)} = 5$.

For $x < 1$, $|x-1| = -(x-1)$, and $f(1^-) = \lim_{x\to 1^-} \frac{-4(x-1)+(x-1)}{-(x-1)} = \lim_{x\to 1^-} \frac{3(x-1)}{(x-1)} = 3$.

Thus $f(1^-) \neq f(1^+)$ and hence the limit does not exist.

EXERCISE 9.1

In problems 1-6, using the table estimate the value of the limit.

(1) $\lim_{x\to 2} \frac{x-2}{x^2-x-2}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.344820	0.33444	0.33344	0.333222	0.33222	0.332258

(2) $\lim_{x\to 2} \frac{x-2}{x^2-4}$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.25641	0.25062	0.250062	0.24993	0.24937	0.24390

(3) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3}}{x}$

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
f(x)	0.2911	0.2891	0.2886	0.2886	0.2885	0.28631

(4) $\lim_{x\to -3} \frac{\sqrt{1-x}-2}{x+3}$

x	– 3.1	– 3.01	– 3.00	– 2.999	– 2.99	– 2.9
$f(x)$	– 0.24845	– 0.24984	– 0.24998	– 0.25001	– 0.25015	– 0.25158

(5) $\lim_{x\to 0} \frac{\sin x}{x}$

x	– 0.1	– 0.01	– 0.001	0.001	0.01	0.1
$f(x)$	0.99833	0.99998	0.99999	0.99999	0.99998	0.99833

(6) $\lim_{x\to 0} \frac{\cos x-1}{x}$

x	– 0.1	– 0.01	– 0.001	0.0001	0.01	0.1
$f(x)$	0.04995	0.0049999	0.0004999	–0.0004999	–0.004999	–0.04995

In exercise problems 7 – 15, use the graph to find the limits (if it exists). If the limit does not exist, explain why?

(7) $\lim\limits_{x\to 3}(4-x)$.

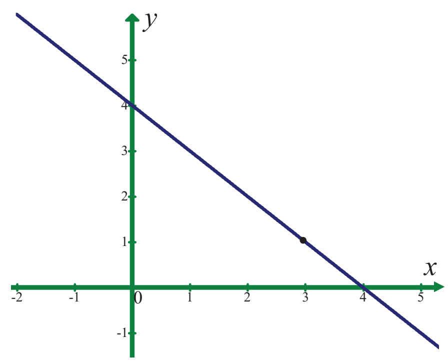


Fig. 9.13

(8) $\lim\limits_{x\to 1}(x^2+2)$.

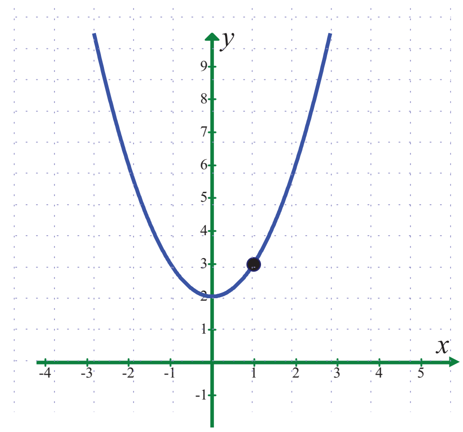


Fig. 9.14

(9) $\lim\limits_{x\to 2} f(x)$

where $f(x) = \begin{cases} 4-x, & x \neq 2 \\ 0, & x = 2 \end{cases}$.

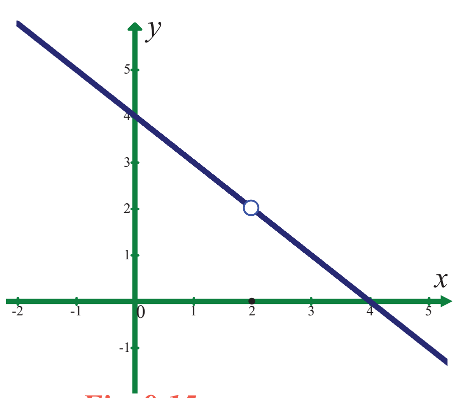


Fig. 9.15

(10) $\lim\limits_{x\to 1} f(x)$

where $f(x) = \begin{cases} x^2+2, & x \neq 1 \\ 1, & x = 1 \end{cases}$.

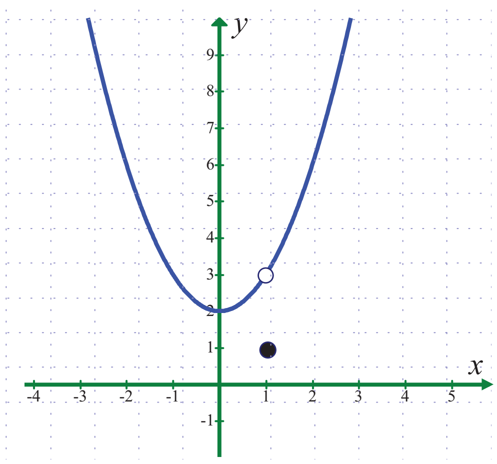


Fig. 9.16

(11) $\lim\limits_{x\to 3}\dfrac{1}{x-3}$

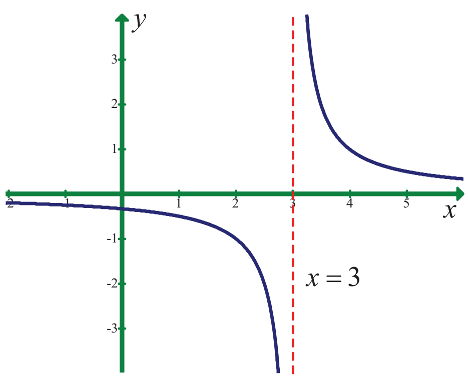


Fig. 9.17

(12) $\lim\limits_{x\to 5}\dfrac{|x-5|}{x-5}$

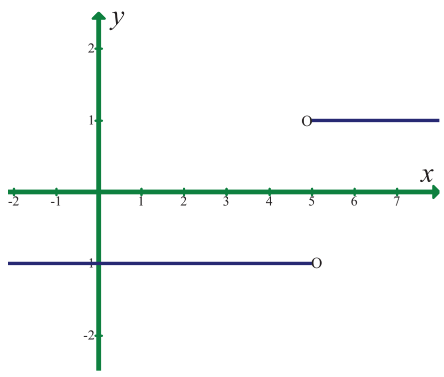


Fig. 9.18

(13) $\lim\limits_{x \to 1} \sin \pi x$

(14) $\lim\limits_{x \to 0} \sec x$

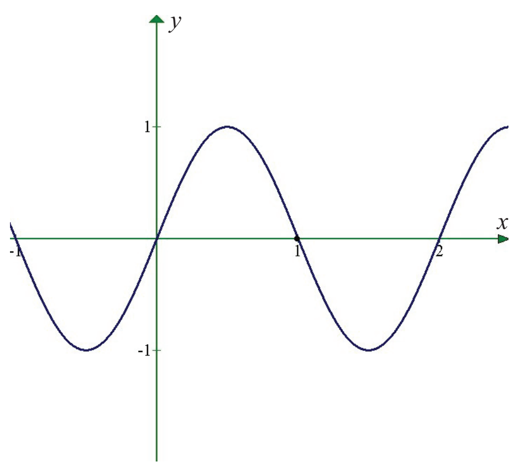


Fig. 9.19

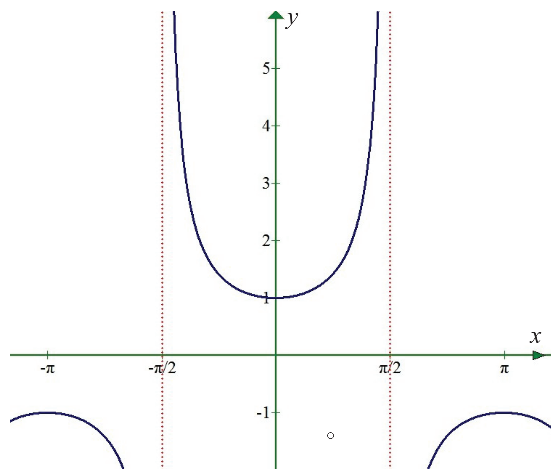


Fig. 9.20

(15) $\lim\limits_{x \to \frac{\pi}{2}} \tan x$

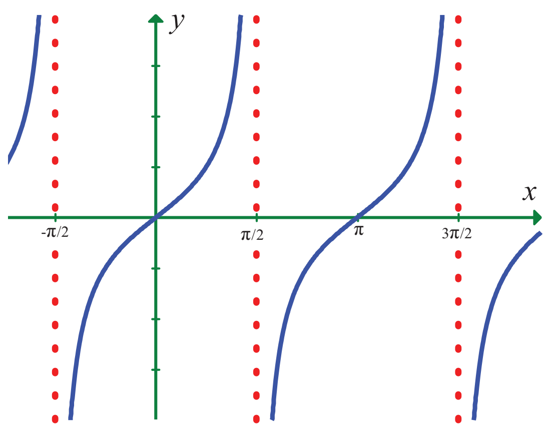


Fig. 9.21

Sketch the graph of f, then identify the values of x_0 for which $\lim\limits_{x \to x_0} f(x)$ exists.

(16) $f(x) = \begin{cases} x^2, & x \le 2 \\ 8 - 2x, & 2 < x < 4 \\ 4, & x \ge 4 \end{cases}$.

(17) $f(x) = \begin{cases} \sin x, & x < 0 \\ 1 - \cos x, & 0 \le x \le \pi \\ \cos x, & x > \pi \end{cases}$

(18) Sketch the graph of a function f that satisfies the given values :

(i) $f(0)$ is undefined

$\lim\limits_{x \to 0} f(x) = 4$

$f(2) = 6$

$\lim\limits_{x \to 2} f(x) = 3$

(ii) $f(-2) = 0$

$f(2) = 0$

$\lim\limits_{x \to -2} f(x) = 0$

$\lim\limits_{x \to 2} f(x)$ does not exist.

(19) Write a brief description of the meaning of the notation $\lim_{x\to 8} f(x) = 25$.

(20) If $f(2) = 4$, can you conclude anything about the limit of $f(x)$ as x approaches 2?

(21) If the limit of $f(x)$ as x approaches 2 is 4, can you conclude anything about $f(2)$? Explain reasoning.

(22) Evaluate : $\lim_{x\to 3} \frac{x^2-9}{x-3}$ if it exists by finding $f(3^-)$ and $f(3^+)$.

(23) Verify the existence of $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} \frac{|x-1|}{x-1}, & \text{for } x \neq 1 \\ 0, & \text{for } x = 1 \end{cases}$.

9.2.3 Theorems on limits

The intention of the informal discussion in the earlier section was to have an intuitive grasp of existence or non existence of the limit. However, it is neither desirable nor practical in every instance, to reach a conclusion about the existence of a limit based on a graph or table of functional values. We must be able to evaluate a limit, or discern its non existence, in a somewhat mechanical fashion. The theorems that we shall consider in this section establish such a means. The proofs of these theorems are more of technical and are beyond the scope of this textbook.

In Illustration 9.1, we concluded that $\lim_{x\to 2}(x^2+3) = 2^2+3 = 7$. That is, the limit of $f(x) = x^2 + 3$ as x tends to 2 is equal to $f(x)$ evaluated at $x = 2$. [That is, $f(2)$]. However, this process of evaluation, as noted earlier, will not always work because $f(x)$ may not even be defined at x_0. Nevertheless, it is true that **if f is a polynomial, then it is always possible to calculate the limit by evaluation**.

Theorem 9.1

Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_n x^n$ be a polynomial, where $a_0, a_1, \cdots, a_n$ are real numbers and n is a fixed positive integer. Then

$$\lim_{x\to x_0} P(x) = a_0 + a_1x_0 + a_2x_0^2 + \cdots + a_nx_0^n = P(x_0).$$

Example 9.7

Calculate $\lim_{x\to 3}(x^3 - 2x + 6)$.

Solution

$P(x) = x^3 - 2x + 6$ is a polynomial.

Hence, $\lim_{x\to 3} P(x) = P(3) = 3^3 - 2\times 3 + 6 = 27$.

Example 9.8

Calculate $\lim_{x\to x_0}(5)$ for any real number x_0.

Solution

$f(x) = 5$ is a polynomial (of degree 0).

Hence $\lim_{x\to x_0}(5) = f(x_0) = 5$.

The limit of a constant function is that constant.

Theorem 9.2

Let I be an open interval containing $x_0 \in \mathbb{R}$.

Let $f, g : I \to \mathbb{R}$.

Suppose that c is a constant and the limits $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ exist. Then $\lim\limits_{x \to x_0} (cf(x)), \lim\limits_{x \to x_0} [f(x) + g(x)]$, $\lim\limits_{x \to x_0} [f(x) - g(x)]$, $\lim\limits_{x \to x_0} [f(x)g(x)]$ and $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}, g(x) \neq 0$, all exist. Moreover,

(i) $\lim\limits_{x \to x_0} cf(x) = c \lim\limits_{x \to x_0} f(x)$,

(ii) $\lim\limits_{x \to x_0} [f(x) \pm g(x)] = \lim\limits_{x \to x_0} f(x) \pm \lim\limits_{x \to x_0} g(x)$,

(iii) $\lim\limits_{x \to x_0} [f(x).g(x)] = \lim\limits_{x \to x_0} f(x) . \lim\limits_{x \to x_0} g(x)$ and

(iv) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$, provided $\lim\limits_{x \to x_0} g(x) \neq 0$.

These results can be extended to any finite number of functions. ◻

Example 9.9

Compute (i) : $\lim\limits_{x \to 8} (5x)$ (ii) $\lim\limits_{x \to -2} \left(-\dfrac{3}{2} x \right)$.

Solution

(i) $\lim\limits_{x \to 8} (5x) = 5 \lim\limits_{x \to 8} (x) = 5 \times 8 = 40$.

(ii) $\lim\limits_{x \to -2} \left(-\dfrac{3}{2} x \right) = -\dfrac{3}{2} \lim\limits_{x \to -2} (x) = \left(-\dfrac{3}{2} \right) (-2) = 3$.

Example 9.10

Compute $\lim\limits_{x \to 0} \left[\dfrac{x^2 + x}{x} + 4x^3 + 3 \right]$.

Solution

$$\begin{aligned} \lim_{x \to 0} \left[\frac{x^2 + x}{x} + 4x^3 + 3 \right] &= \lim_{x \to 0} \left(\frac{x^2 + x}{x} \right) + \lim_{x \to 0} (4x^3 + 3) \\ &= \lim_{x \to 0} (x + 1) + \lim_{x \to 0} (4x^3 + 3) \\ &= (0 + 1) + (0 + 3) \\ &= 4. \end{aligned}$$

Example 9.11

Calculate $\lim\limits_{x \to -1} (x^2 - 3)^{10}$.

Solution

$$\lim_{x \to -1} (x^2 - 3) = 1 - 3 = -2.$$

$$\text{Therefore, } \lim_{x \to -1} (x^2 - 3)^{10} = \lim_{x \to -1} (x^2 - 3)\,(x^2 - 3)\ldots(x^2 - 3) \text{ (10 times)}$$

$$= \lim_{x \to -1} (x^2 - 3) \lim_{x \to -1} (x^2 - 3)\ldots \lim_{x \to -1} (x^2 - 3) \text{ (10 times)}$$

$$= \left[\lim_{x \to -1} (x^2 - 3)\right]^{10} = (-2)^{10} = 2^{10} = 1024.$$

Note that $\lim\limits_{x \to -1} (x^2 - 3)^{10} = \left[\lim\limits_{x \to -1} (x^2 - 3)\right]^{10}$.

Theorem 9.3

If $\lim\limits_{x \to x_0} f(x)$ exists then $\lim\limits_{x \to x_0} [f(x)]^n$ exists and $\lim\limits_{x \to x_0} [f(x)]^n = \left[\lim\limits_{x \to x_0} f(x)\right]^n$.

Example 9.12

Calculate $\lim\limits_{x \to -2} (x^3 - 3x + 6)\,(-x^2 + 15)$.

Solution

$$\lim_{x \to -2} (x^3 - 3x + 6) = (-2)^3 - 3(-2) + 6 = -8 + 6 + 6 = 4$$

$$\lim_{x \to -2} (-x^2 + 15) = -(-2)^2 + 15 = -4 + 15 = 11$$

$$\lim_{x \to -2} (x^3 - 3x + 6)\,(-x^2 + 15) = \lim_{x \to -2} (x^3 - 3x + 6) \lim_{x \to -2} (-x^2 + 15) = 4 \times 11 = 44.$$

Example 9.13

Calculate $\lim\limits_{x \to 3} \dfrac{(x^2 - 6x + 5)}{x^3 - 8x + 7}$.

Solution

$$\lim_{x \to 3} (x^2 - 6x + 5) = 3^2 - 6 \times 3 + 5 = -4$$

$$\lim_{x \to 3} (x^3 - 8x + 7) = 3^3 - 8 \times 3 + 7 = 10 \neq 0.$$

$$\text{Therefore, } \lim_{x \to 3} \frac{(x^2 - 6x + 5)}{x^3 - 8x + 7} = \frac{\lim\limits_{x \to 3} (x^2 - 6x + 5)}{\lim\limits_{x \to 3} (x^3 - 8x + 7)} = \frac{-4}{10} = -\frac{2}{5}.$$

Caution

Do not use the limit theorem for the quotient if $\lim\limits_{x \to x_0} g(x) = 0$.

Example 9.14

Compute $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$.

Solution

Here $\lim_{x \to 1}(x-1) = 0$. In such cases, rationalise the numerator.

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1} \frac{(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{\lim_{x \to 1}(1)}{\lim_{x \to 1}\left(\sqrt{x}+1\right)} = \frac{1}{2}.$$

Example 9.15

Find $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

Solution

We can't apply the quotient theorem immediately. Use the algebra technique of rationalising the numerator.

$$\frac{\sqrt{t^2+9}-3}{t^2} = \frac{\left(\sqrt{t^2+9}-3\right)\left(\sqrt{t^2+9}+3\right)}{t^2\left(\sqrt{t^2+9}+3\right)} = \frac{t^2+9-9}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t \to 0} \frac{t^2}{t^2\sqrt{t^2+9}+3} = \lim_{t \to 0} \frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

Theorem 9.4

$$\lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}.$$

Proof

We know that $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})$

$$\begin{aligned}
\lim_{x \to a} \frac{x^n - a^n}{x-a} &= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{(x-a)} \\
&= \lim_{x \to a}(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-1} + \cdots + a^{n-1} \quad (n \text{ times}) \\
\lim_{x \to a} \frac{x^n - a^n}{x-a} &= na^{n-1}.
\end{aligned}$$

It is also true for any rational number n.

Example 9.16

Compute $\lim_{x \to 1} \frac{x^3-1}{x-1}$.

Solution

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{x^3-1^3}{x-1} = 3(1)^{3-1} = 3.$$

Example 9.17

Calculate $\lim\limits_{t\to 1}\frac{\sqrt{t}-1}{t-1}$.

Solution

$$\lim_{x\to 1}\frac{\sqrt{t}-1}{t-1} = \lim_{x\to 1}\frac{t^{\frac{1}{2}}-1^{\frac{1}{2}}}{t-1} = \frac{1}{2}(1)^{\frac{1}{2}-1} = \frac{1}{2}.$$

Example 9.18

Find $\lim\limits_{x\to 0}\frac{(2+x)^5-2^5}{x}$.

Solution

Put $2 + x = y$ so that as $y \to 2$ as $x \to 0$.

Therefore, $\lim\limits_{x\to 0}\frac{(2+x)^5-2^5}{x} = \lim\limits_{y\to 2}\frac{y^5-2^5}{y-2} = 5(2^4) = 80$.

Example 9.19

Find the positive integer n so that $\lim\limits_{x\to 3}\frac{x^n-3^n}{x-3} = 27$.

Solution

$$\lim_{x\to 3}\frac{x^n-3^n}{x-3} = n.3^{n-1} = 27$$

That is $n.3^{n-1} = 3\times 3^2 = 3\times 3^{3-1} \Rightarrow n = 3$.

Example 9.20

Find the relation between a and b if $\lim\limits_{x\to 3} f(x)$ exists where $f(x) = \begin{cases} ax+b & \text{if } x>3 \\ 3ax-4b+1 & \text{if } x<3 \end{cases}$.

Solution

$\lim\limits_{x\to 3^-} f(x) = 9a - 4b + 1$

$\lim\limits_{x\to 3^+} f(x) = 3a+b$. Now the existence of limit forces us to have

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^+} f(x)$.

$\Rightarrow 9a - 4b + 1 = 3a + b$

$\Rightarrow 6a - 5b + 1 = 0.$

EXERCISE 9.2

Evaluate the following limits :

(1) $\lim\limits_{x\to 2}\frac{x^4-16}{x-2}$

(2) $\lim\limits_{x\to 1}\frac{x^m-1}{x^n-1}$, m and n are integers.

(3) $\lim\limits_{\sqrt{x}\to 3}\frac{x^2-81}{\sqrt{x}-3}$

(4) $\lim\limits_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}, x>0$

(5) $\lim\limits_{x\to 5}\frac{\sqrt{x+4}-3}{x-5}$

(6) $\lim\limits_{x\to 2}\dfrac{\frac{1}{x}-\frac{1}{2}}{x-2}$ (7) $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-x^2}{1-\sqrt{x}}$ (8) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+16}-4}$

(9) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$ (10) $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{7+x^3}-\sqrt{3+x^2}}{x-1}$ (11) $\lim\limits_{x\to 2}\dfrac{2-\sqrt{x+2}}{\sqrt[3]{2}-\sqrt[3]{4-x}}$

(12) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x}$ (13) $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x}-1}{x^2}$ (14) $\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$

(15) $\lim\limits_{x\to a}\dfrac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}(a>b)$

9.2.4 Infinite limits and limits at infinity

Infinite Limits

BBS1KB

Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be defined by $f(x)=\dfrac{1}{x^2}$.

Let us consider the problem of calculating $\lim\limits_{x\to 0}\dfrac{1}{x^2}$.

The following table gives the values of $\dfrac{1}{x^2}$ near 0.

$x \to 0^+$			$x \to 0^-$		
x	x^2	$\frac{1}{x^2}$	x	x^2	$\frac{1}{x^2}$
1	1	1	–1	1	1
0.5	0.25	4	– 0.5	0.25	4
0.1	0.01	100	– 0.1	0.01	100
0.01	0.0001	10,000	– .01	0.0001	10,000
0.001	0.000001	10,00,000	– 0.001	0.000001	10,00,000
0.0001	0.00000001	10,00,00,000	– 0.0001	0.00000001	10,00,00,000

The table values tell us that as x gets closer and closer to 0, $f(x)=\dfrac{1}{x^2}$ gets larger and larger. In fact, $\dfrac{1}{x^2}$ grows without bound as x approaches 0 from either side. In this situation we say that $f(x)$ **tends to infinity as x approaches zero** and write $\dfrac{1}{x^2}\to\infty$ as $x\to 0^-$ and $\dfrac{1}{x^2}\to\infty$ as $x\to 0^+$

and hence $\dfrac{1}{x^2}\to\infty$ as $x\to 0$.

Geometrically, $x = 0$ namely the y-axis is a vertical asymptote to the curve representing $f(x) = \frac{1}{x^2}$.

The graph of the function $f(x) = \frac{1}{x^2}$ is shown in Fig. 9.22.

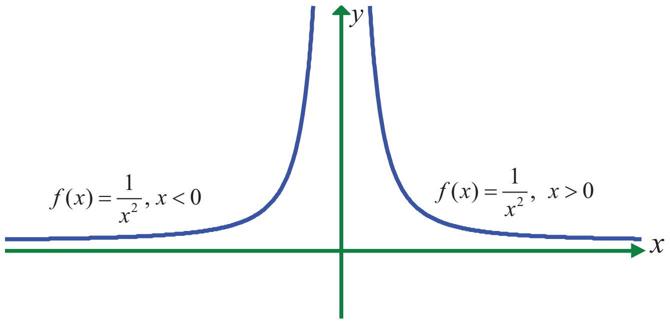


Fig. 9.22

Remember that the limit is infinite and so $\lim\limits_{x\to 0}\frac{1}{x^2}$ does not exist. Students are cautioned that ∞ is a symbol for this behaviour of $f(x) = \frac{1}{x^2}$ and is not a new number.

Similarly, if we look at $f(x) = \frac{1}{x}$,

it is easy to see that, $\frac{1}{x} \to -\infty$ as $x \to 0^-$

and $\frac{1}{x} \to +\infty$ as $x \to 0^+$

which is geometrically clear from the graph of $f(x) = \frac{1}{x}$ (Fig. 9.23) .

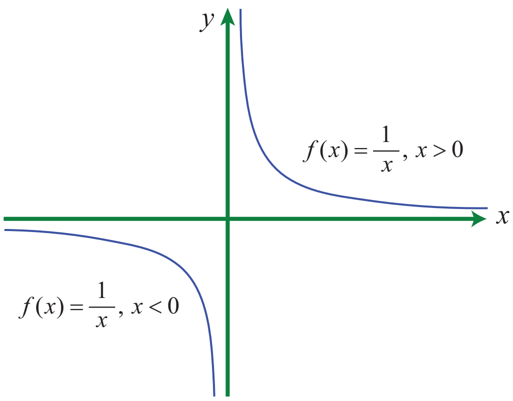


Fig. 9.23

In general we have the following intuitive definitions.

Definition 9.4

For a given $M > 0$, open intervals of the form (M, ∞) is called neighbourhood of ∞.

Similarly, for given $K < 0$, open intervals of the form $(-\infty, K)$ is called neighbourhood of $-\infty$.

Definition 9.5

We say, $f(x) \to \infty$ as x approaches x_0 if for given positive number M there is a neighbourhood of x_0, such that whenever x is in the neighbourhood of x_0, $f(x) > M$. i.e., $f(x) \in (M, \infty)$.

Similarly, $f(x) \to -\infty$ as x approaches x_0 if for a given $K < 0$ there is a neighbourhood of x_0 such that whenever x is in the neighbourhood of x_0, $f(x) < K$. i.e., $f(x) \in (-\infty, K)$.

To describe this situation symbolically, we write

$f(x) \to \infty$ as $x \to x_0$

$f(x) \to -\infty$ as $x \to x_0$

$f(x) \to \infty$ as $x \to {x_0}^-$

$f(x) \to -\infty$ as $x \to {x_0}^+$

and $f(x) \to \infty$ as $x \to {x_0}^+$

$f(x) \to -\infty$ as $x \to {x_0}^-$

are called **infinite limits**. If any one of the foregoing conditions hold, then the line $x = x_0$ is a **vertical asymptote** for the graph of $f(x)$.

Example 9.21

Calculate $\lim_{x\to 0}\frac{1}{(x^2+x^3)}$.

Solution

One can tabulate values of x near 0 (from either side) and conclude $f(x)=\frac{1}{x^2+x^3}$ grows without bound and hence $f(x)\to\infty$ as $x\to 0$.

To calculate this limit without making a table, we first divide the numerator and denominator by x^2. This division can be done, since in the calculation of the limit $x\neq 0$ and hence $x^2\neq 0$. We can have

$$\lim_{x\to 0}\frac{1}{x^2+x^3}=\lim_{x\to 0}\frac{\frac{1}{x^2}}{\frac{x^2+x^3}{x^2}}=\frac{\lim_{x\to 0}\left(\frac{1}{x^2}\right)}{\lim_{x\to 0}(1+x)}.$$

Now $\frac{1}{x^2}\to\infty$ as $x\to 0$ and $\lim_{x\to 0}(1+x)=1$.

Thus the numerator grows without bound while the denominator approaches 1, implying that $\frac{1}{(x^2+x^3)}$ does tend to infinity.

This example illustrates how a difficult calculation can be greatly simplified by a few algebraic manipulations.

Example 9.22

Evaluate $\lim_{x\to 2}\frac{1}{(x-2)^3}$.

Solution

From the graph of $f(x)=\frac{1}{(x-2)^3}$,

clearly, $\frac{1}{(x-2)^3}\to-\infty$ as $x\to 2^-$ and

$\frac{1}{(x-2)^3}\to\infty$ as $x\to 2^+$.

Hence the limit does not exist.

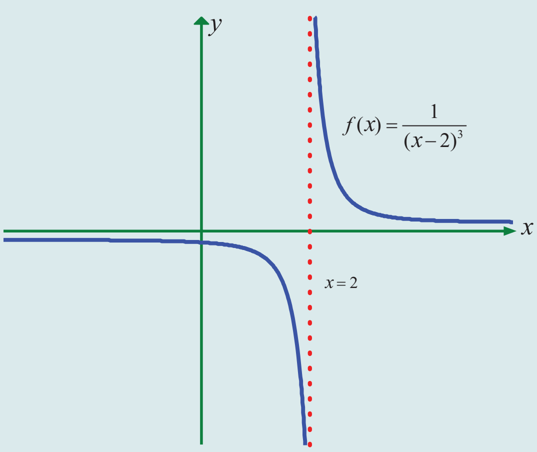


Fig. 9.24

In general

(i) If n is an even positive integer then

$$\frac{1}{(x-a)^n}\to\infty \text{ as } x\to a$$
$$\frac{1}{(x-a)^n}\to\infty \text{ as } x\to a^-$$
$$\frac{1}{(x-a)^n}\to\infty \text{ as } x\to a^+$$

(ii) If n is an odd positive integer, then

$$\frac{1}{(x-a)^n}\to-\infty \text{ as } x\to a^-$$
$$\frac{1}{(x-a)^n}\to+\infty \text{ as } x\to a^+.$$

The line $x=a$ becomes a vertical asymptote.

9.2.5 Limits at infinity

In the previous section we investigated infinite limits and vertical asymptotes. There, we let x approach a number and the result was that the values of y became arbitrarily large (very large positive or very large negative). In this section, we let x become arbitrarily large (positive or negative) and see what happens to y.

Let's begin by investigating the behaviour of $f:\mathbb{R}\to\mathbb{R}$ defined by

$f(x) = \dfrac{x^2-1}{x^2+1}$ as x becomes large.

We tabulate the values of this function

x	$f(x)$
0	-1
±1	0
±2	0.6000000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

As x grows larger and larger (large positive or large negative) you can see that the values of $f(x)$ gets closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x\to\pm\infty}\frac{x^2-1}{x^2+1} = 1.$$

If we look at the graph :

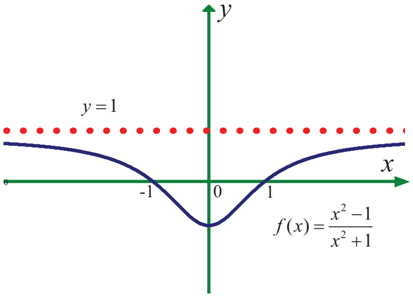


Fig. 9.25

Geometrically, (see Fig. 9.25) this situation also leads us to have

Definition 9.6

The line $y = l$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to-\infty} f(x) = l \quad\text{or}\quad \lim_{x\to+\infty} f(x) = l.$$

Illustration 9.4

If $f:\mathbb{R}\to\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ is defined by $f(x)=\tan^{-1}x$,

find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

Solution

If we look at the graph of $y = \tan^{-1}x$,

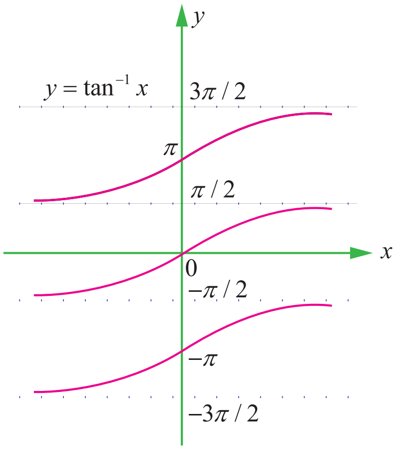


Fig. 9.26

$$\lim_{x\to-\infty}\tan^{-1}x = -\frac{\pi}{2},$$

$$\lim_{x\to+\infty}\tan^{-1}x = \frac{\pi}{2}.$$

Illustration 9.5

Calculate $\lim\limits_{x\to\infty}\dfrac{2x^2-2x+3}{x^2+4x+4}$.

Solution

If we could try to use limit theorems to calculate this limit, we end up in the following situations.

$(2x^2-2x+3)\to\infty$ as $x\to\infty$

$(x^2+4x+3)\to\infty$ as $x\to\infty$

$\lim\limits_{x\to\infty}\left(\dfrac{2x^2-2x+3}{x^2+4x+3}\right)=\dfrac{\infty}{\infty}$ which is called an indeterminate form.

But actual calculation and tabulation gives the following :

x	$2x^2-2x+3$	x^2+4x+4	$\dfrac{2x^2-2x+3}{x^2+4x+4}$
1	3	9	0.3333
10	183	144	1.27083
100	19803	10404	1.90340
1000	1998003	1004004	1.99003
10000	199980003	100040004	1.99900

Table values show that as x becomes sufficiently large, $f(x)$ becomes closer and closer to 2. Then,

$\lim\limits_{x\to\infty}\left(\dfrac{2x^2-2x+3}{x^2+4x+4}\right)=2.$

Fortunately, we may simplify the problem by dividing the numerator and denominator by x^2. We have

$$\lim_{x\to\infty}\frac{2x^2-2x+3}{x^2+4x+4}=\lim_{x\to\infty}\left(\frac{2-\frac{2}{x}+\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}}\right)$$

$$=\frac{2-0+0}{1+0+0}\quad\left(\text{since }\frac{1}{x}\to 0\text{ as }x\to\infty,\ \frac{1}{x^2}\to 0\text{ as }x\to\infty\right)$$

$$=2.$$

Note that the degree of both numerator and denominator expressions are the same.

In general, the limits as $x\to\pm\infty$ of rational expressions can be found by first dividing the numerator and denominator by the highest power of x that appears in the denominator, and then calculating the limit as $x\to\infty$ (or $x\to-\infty$) of both numerator and denominator.

Example 9.23

Calculate $\lim\limits_{x\to\infty}\dfrac{x^3+2x+3}{(5x^2+1)}$.

Solution

Dividing by x^2

$$\lim_{x\to\infty}\frac{x^3+2x+3}{(5x^2+1)} = \lim_{x\to\infty}\frac{x+\frac{2}{x}+\frac{3}{x^2}}{5+\frac{1}{x^2}} \to \infty$$

That is, $\frac{x^3+2x+3}{(5x^2+1)} \to \infty$ as $x\to\infty$.

In other words, the limit does not exist.

Note that the degree of numerator is higher than that of the denominator.

Example 9.24

Calculate $\lim_{x\to\infty}\frac{1-x^3}{3x+2}$.

Solution

Dividing by x, we get

$$\frac{1-x^3}{3x+2} = \frac{\frac{1}{x}-x^2}{3+\frac{2}{x}} \to -\infty \text{ as } x\to\infty.$$

Therefore the limit does not exist.

9.2.6 Limits of rational functions

If $R(x) = \frac{p(x)}{q(x)}$ and the degree of the polynomial $p(x)$ is greater than the degree of $q(x)$, then

$\frac{p(x)}{q(x)} \to +\infty$ or $-\infty$ as $x\to\infty$.

If the degree of $q(x)$ is greater than the degree of $p(x)$, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = 0.$$

Finally, if the degree of $p(x)$ is equal to the degree of $q(x)$, then

$$\lim_{x\to\infty}\frac{p(x)}{q(x)} = \frac{\text{coeffiecent of highest power of } x \text{ in } p(x)}{\text{coefficient of highest power of } x \text{ in } q(x)}.$$

Remark

We reemphasize that statements such as $f(x)\to\infty$ as $x\to a$, $f(x)\to-\infty$ as $x\to a$, and $f(x)\to\infty$ as $x\to\infty$, $f(x)\to-\infty$ as $x\to\infty$ mean that the limits do not exist. The symbol ∞ does not represent a number and should not be treated as a number.

9.2.7 Applications of limits

Example 9.25

Alcohol is removed from the body by the lungs, the kidneys, and by chemical processes in liver. At moderate concentration levels, the majority work of removing the alcohol is done by the liver; less than 5% of the alcohol is eliminated by the lungs and kidneys. The rate r at which the liver processes alcohol from the bloodstream is related to the blood alcohol concentration x by a rational function of the form $r(x) = \dfrac{\alpha x}{x+\beta}$ for some positive constants α and β. Find the maximum possible rate of removal.

Solution

As the alcohol concentration x increases the rate of removal increases.

Therefore, the maximum possible rate of removal $= \lim\limits_{x\to\infty} r(x)$

$$= \lim_{x\to\infty} \frac{\alpha x}{x+\beta} = \lim_{x\to\infty} \frac{\alpha}{\left(1+\dfrac{\beta}{x}\right)} = \alpha.$$

9.2.8 Sandwich Theorem

Sandwich theorem is also known as squeeze theorem. As shown in the figure 9.27, if $f(x)$ is 'squeezed' or 'sandwiched' between $g(x)$ and $h(x)$ for all x close to x_0, and if we know that the functions g and h have a common limit l as $x \to x_0$, it stands to reason that f also approaches l as $x \to x_0$.

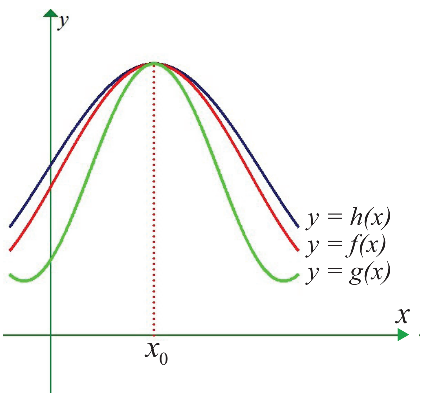


Fig. 9.27

Theorem 9.5 (Sandwich Theorem)

If $f, g, h : I \subseteq \mathbb{R} \to \mathbb{R}$ such that $g(x) \le f(x) \le h(x)$ for all x in a deleted neighbourhood of x_0 contained in I, and if

$$\lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = l, \quad \text{then} \quad \lim_{x\to x_0} f(x) = l.$$

Example 9.26

According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = \dfrac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$. Why is a left hand limit necessary?

Solution

$$\lim_{v\to c^-}(m) = \lim_{v\to c^-} \frac{m_0}{\sqrt{1-\dfrac{v^2}{c^2}}} = \frac{m_0}{\sqrt{\lim\limits_{v\to c^-}\left(1-\dfrac{v^2}{c^2}\right)}}$$

For $h > 0$, $c - h < v < c$. This implies, $(c-h)^2 < v^2 < c^2$.

That is, $\dfrac{(c-h)^2}{c^2} < \dfrac{v^2}{c^2} < 1$. That is, $\lim\limits_{h\to 0}\dfrac{(c-h)^2}{c^2} < \lim\limits_{h\to 0}\dfrac{v^2}{c^2} < \lim\limits_{h\to 0} 1$.

That is, $1 < \lim\limits_{h \to 0} \frac{v^2}{c^2} < 1$. That is, $1 < \lim\limits_{v \to c^-} \frac{v^2}{c^2} < 1$. By Sandwich theorem, $\lim\limits_{v \to c^-} = 1$.

Therefore, $\lim\limits_{v \to c^-} (m) \to \infty$.

That is, the mass becomes very very large (infinite) as $v \to c^-$.

The left hand limit is necessary. Otherwise as $v \to c^+$ makes $1 - \frac{v^2}{c^2} < 0$ and consequently we cannot find the mass.

Example 9.27

The velocity in ft/sec of a falling object is modeled by $r(t) = -\sqrt{\frac{32}{k}} \frac{1 - e^{-2t\sqrt{32k}}}{1 + e^{-2t\sqrt{32k}}}$, where k is a constant that depends upon the size and shape of the object and the density of the air. Find the limiting velocity of the object, that is, find $\lim\limits_{t \to \infty} r(t)$.

Solution

$$\begin{aligned}
\lim_{t \to \infty} r(t) &= \lim_{t \to \infty} -\sqrt{\frac{32}{k}} \frac{1 - e^{-2t\sqrt{32k}}}{1 + e^{-2t\sqrt{32k}}} \\
&= -\sqrt{\frac{32}{k}} \lim_{t \to \infty} \frac{1 - e^{-2t\sqrt{32k}}}{1 + e^{-2t\sqrt{32k}}} \\
&= -\sqrt{\frac{32}{k}} \frac{(1-0)}{(1+0)} = -\sqrt{\frac{32}{k}} \text{ ft/sec.}
\end{aligned}$$

Example 9.28

Suppose that the diameter of an animal's pupils is given by $f(x) = \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15}$, where x is the intensity of light and $f(x)$ is in *mm*. Find the diameter of the pupils with (a) minimum light (b) maximum light.

Solution

(a) For minimum light it is enough to find the limit of the function when $x \to 0^+$.

$$\begin{aligned}
\lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} = \lim_{x \to 0^+} \frac{160 + 90x^{0.4}}{4 + 15x^{0.4}} \\
&= \frac{160}{4} = 40\text{mm}.
\end{aligned}$$

(b) For maximum light, it is enough to find the limit of the function when $x \to \infty$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} = \frac{90}{15} = 6\text{mm}$$

That is, the pupils have a limiting size of 6mm, as the intensity of light is very large.

EXERCISE 9.3

(1) (a) Find the left and right limits of $f(x)=\dfrac{x^2-4}{(x^2+4x+4)(x+3)}$ at $x=-2$.

(b) $f(x)=\tan x$ at $x=\dfrac{\pi}{2}$.

Evaluate the following limits

(2) $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2(x^2-6x+9)}$

(3) $\lim\limits_{x\to\infty}\dfrac{3}{x-2}-\dfrac{2x+11}{x^2+x-6}$

(4) $\lim\limits_{x\to\infty}\dfrac{x^3+x}{x^4-3x^2+1}$

(5) $\lim\limits_{x\to\infty}\dfrac{x^4-5x}{x^2-3x+1}$

(6) $\lim\limits_{x\to\infty}\dfrac{1+x-3x^3}{1+x^2+3x^3}$

(7) $\lim\limits_{x\to\infty}\left(\dfrac{x^3}{2x^2-1}-\dfrac{x^2}{2x+1}\right)$

(8) Show that

(i) $\lim\limits_{n\to\infty}\dfrac{1+2+3+\ldots+n}{3n^2+7n+2}=\dfrac{1}{6}$

(ii) $\lim\limits_{n\to\infty}\dfrac{1^2+2^2+\ldots+(3n)^2}{(1+2+\ldots+5n)(2n+3)}=\dfrac{9}{25}$

(iii) $\lim\limits_{n\to\infty}\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\ldots+\dfrac{1}{n(n+1)}=1$

(9) An important problem in fishery science is to estimate the number of fish presently spawning in streams and use this information to predict the number of mature fish or "recruits" that will return to the rivers during the reproductive period. If S is the number of spawners and R the number of recruits, "Beverton-Holt spawner recruit function" is $R(S)=\dfrac{S}{(\alpha S+\beta)}$ where α and β are positive constants. Show that this function predicts approximately constant recruitment when the number of spawners is sufficiently large.

(10) A tank contains 5000 litres of pure water. Brine (very salty water) that contains 30 grams of salt per litre of water is pumped into the tank at a rate of 25 litres per minute. The concentration of salt water after t minutes (in grams per litre) is $C(t)=\dfrac{30t}{200+t}$.
What happens to the concentration as $t\to\infty$?

Example 9.29

Evaluate $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{1}{x}\right)$.

Solution

We know that $-1\le\sin\dfrac{1}{x}\le 1\Rightarrow -x^2\le x^2\sin\dfrac{1}{x}\le x^2$

$$\text{Take } g(x)=-x^2,\ f(x)=x^2\sin\frac{1}{x};\ h(x)=x^2$$

$$\text{Then } \lim_{x\to 0}g(x)=\lim_{x\to 0}(-x^2)=0 \text{ and}$$

$$\lim_{x\to 0} h(x) = \lim_{x\to 0}(x^2) = 0.$$

By Sandwich theorem,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

We could have wrongly concluded had we resorted to applying the limit theorems, namely

$$\lim_{x\to 0} x^2\left(\sin\frac{1}{x}\right) = \lim_{x\to 0}(x^2)\lim_{x\to 0}\left(\sin\frac{1}{x}\right)$$

Now, $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist, $\lim_{x\to 0} x^2 = 0$ and hence $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$ leading us into trouble.

Note that, if $a \le f(x) \le a$, then $\lim_{x\to x_0} f(x) = a$.

Example 9.30

Prove that $\lim_{x\to 0}\sin x = 0$.

Solution

Since $-x \le \sin x \le x$ for all $x \ge 0$

$$\lim_{x\to 0}(-x) = 0 \text{ and}$$

$$\lim_{x\to 0}(x) = 0.$$

By Sandwich theorem

$$\lim_{x\to 0}\sin x = 0.$$

Example 9.31

Show that $\lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{15}{x}\right\rfloor\right] = 120$.

Solution

$$\frac{1}{x} - 1 \le \left\lfloor\frac{1}{x}\right\rfloor \le \frac{1}{x} + 1$$

$$\frac{2}{x} - 1 \le \left\lfloor\frac{2}{x}\right\rfloor \le \frac{2}{x} + 1$$

$$\vdots$$

$$\frac{15}{x} - 1 \le \left\lfloor\frac{15}{x}\right\rfloor \le \frac{15}{x} + 1$$

Summing, we get,

$$\frac{120}{x} - 15 \le \left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{15}{x}\right\rfloor \le \frac{120}{x} + 15$$

$$120 - 15x \le x\left[\left\lfloor\frac{1}{x}\right\rfloor + \left\lfloor\frac{2}{x}\right\rfloor + \cdots + \left\lfloor\frac{15}{x}\right\rfloor\right] \le 120 + 15x$$

$$\lim_{x\to 0^+}(120-15x) \le \lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor+\left\lfloor\frac{2}{x}\right\rfloor+\cdots+\left\lfloor\frac{15}{x}\right\rfloor\right] \le \lim_{x\to 0^+}(120+15x)$$

$$120 \le \lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor+\left\lfloor\frac{2}{x}\right\rfloor+\cdots+\left\lfloor\frac{15}{x}\right\rfloor\right] \le 120$$

$$\lim_{x\to 0^+} x\left[\left\lfloor\frac{1}{x}\right\rfloor+\left\lfloor\frac{2}{x}\right\rfloor+\cdots+\left\lfloor\frac{15}{x}\right\rfloor\right] = 120.$$

9.2.9 Two special Trigonometrical limits

Result 9.1

(a) $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ (b) $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}=0$.

Proof

We use a circular sector to prove the result.

Consider the circle with centre (0, 0) and radius 1. Any point on this circle is $P(\cos\theta, \sin\theta)$.

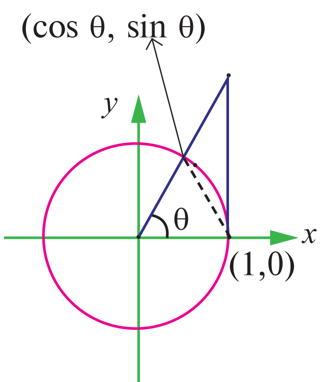


Fig. 9.28

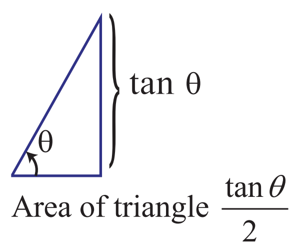


Fig. 9.29

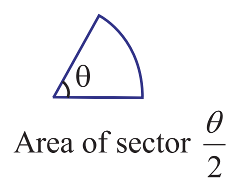


Fig. 9.30

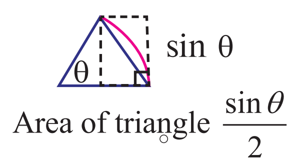


Fig. 9.31

By area property $\dfrac{\tan\theta}{2}\ge\dfrac{\theta}{2}\ge\dfrac{\sin\theta}{2}$.

Multiplying each expression by $\dfrac{2}{\sin\theta}$ produces $\dfrac{1}{\cos\theta}\ge\dfrac{\theta}{\sin\theta}\ge 1$ and taking reciprocals

$\cos\theta\le\dfrac{\sin\theta}{\theta}\le 1$.

Because $\cos(-\theta)=\cos\theta$ and $\dfrac{\sin(-\theta)}{-\theta}=\dfrac{\sin\theta}{\theta}$ one can conclude that this inequality is valid for all non-zero θ in the open interval $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$.

We know that $\lim\limits_{\theta\to 0}\cos\theta=1; \lim\limits_{\theta\to 0}(1)=1$ and applying Sandwich theorem we get $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$. ◻

(b) $\lim\limits_{\theta\to 0}\dfrac{1-\cos\theta}{\theta}=0$.

$$1-\cos\theta = 2\sin^2\frac{\theta}{2}$$

$$\frac{1-\cos\theta}{\theta} = \left(\sin\frac{\theta}{2}\right)\frac{\sin\left(\frac{\theta}{2}\right)}{\left(\frac{\theta}{2}\right)}$$

$$\text{Therefore, } \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\left(\sin\frac{\theta}{2}\right).\lim_{\theta\to 0}\left(\frac{\frac{\sin\theta}{2}}{\frac{\theta}{2}}\right)$$

$$= 0 \times 1 = 0.$$

9.2.10 Some important other limits

Result 9.2

$$\lim_{x\to 0}\frac{e^x-1}{x} = 1.$$

Proof

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots \qquad \text{(from sequences and series)}$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\left(\frac{\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots}{x}\right)$$

$$= \lim_{x\to 0}\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\dots\right) = 1.$$

Result 9.3

$$\lim_{x\to 0}\frac{a^x-1}{x} = \log a, \quad a>0$$

Proof

We know that $a^x, \log_a x$ are inverses of each other.

Since $\log f(x)$ is the inverse of $\exp(f(x))$, $\exp(\log f(x)) = f(x)$.

Therefore, $a^x = \exp(\log a^x)$

$= e^{x\log a}$

$$\text{Therefore, } \frac{a^x-1}{x} = \frac{e^{x\log a}-1}{x\log a}\times\log a$$

Now as $x\to 0,\ y = x\log a \to 0$

$$\text{Therefore, } \lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{e^y-1}{y}\times\log a = \log a\lim_{y\to 0}\left(\frac{e^y-1}{y}\right) = \log a$$

(since $\lim_{x\to 0}\frac{e^x-1}{x} = 1$).

Result 9.4

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=1.$$

Proof

Take $\log(1+x) = y$

Then $y\to 0$ as $x\to 0$ and

$$1+x = e^y$$
$$x = e^y-1$$

Therefore, $\displaystyle\lim_{x\to 0}\frac{\log(1+x)}{x} = \lim_{y\to 0}\frac{y}{e^y-1}$

$$= \lim_{y\to 0}\frac{1}{\left(\frac{e^y-1}{y}\right)}=\frac{1}{1}=1.$$

Some important limits without proof

Results 9.5 to 9.9

(5) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$

(6) $\displaystyle\lim_{x\to 0}\frac{\tan^{-1}x}{x}=1$

(7) $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ exists and this limit is e.

(8) $\displaystyle\lim_{x\to 0}(1+x)^{1/x}=e$

(9) $\displaystyle\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k.$

This number e is also known as transcendental number in the sense that e never satisfies a polynomial (algebraic) equation of the form

$$a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0.$$

Example 9.32

Evaluate : $\displaystyle\lim_{x\to 0}(1+\sin x)^{2\operatorname{cosec} x}$

Solution

Let $\sin x=\dfrac{1}{y}$

As $x\to 0$, $y\to\infty$ and

$$\lim_{x\to 0}(1+\sin x)^{2\operatorname{cosec} x} = \lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{2y} = \left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^2 = e^2.$$

Example 9.33

Evaluate : $\displaystyle\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x.$

Solution

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x = \lim_{x\to\infty}\left(\frac{x-2+4}{x-2}\right)^{x-2+2} = \lim_{x\to\infty}\left(1+\frac{4}{x-2}\right)^{(x-2)+2}$$

Let $y = x-2$, as $x \to \infty$, $y \to \infty$ and (Let $y = x-2$, Then as $x \to \infty$, $y \to \infty$)

$$\lim_{x\to\infty}\left(\frac{x+2}{x-2}\right)^x = \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^{y+2} = \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^y . \lim_{y\to\infty}\left(1+\frac{4}{y}\right)^2$$

$$= e^4 . 1 = e^4 .$$

Example 9.34

Evaluate : $\lim_{x\to\frac{\pi}{4}} \frac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$.

Solution

$$\frac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x} = \frac{2^{\frac{5}{2}}-\left[(\cos x+\sin x)^2\right]^{\frac{5}{2}}}{1-\sin 2x}$$

$$= \frac{2^{\frac{5}{2}}-(1+\sin 2x)^{\frac{5}{2}}}{2-(1+\sin 2x)}$$

$$= \frac{2^{\frac{5}{2}}-(1+\sin 2x)^{\frac{5}{2}}}{2-(1+\sin 2x)}$$

Therefore, $\lim_{x\to\frac{\pi}{4}} \frac{2^{\frac{5}{2}}-[(\cos x+\sin x)^2]^{5/2}}{2-[1+\sin 2x]} = \lim_{x\to\frac{\pi}{4}} \frac{2^{\frac{5}{2}}-[1+\sin 2x]^{\frac{5}{2}}}{2-(1+\sin 2x)}$.

Take $y = 1 + \sin 2x$. As $x \to \frac{\pi}{4}$, $y \to 2$

$$= \lim_{y\to 2} \frac{2^{\frac{5}{2}}-y^{\frac{5}{2}}}{2-y}$$

$$= \frac{5}{2}.2^{\frac{5}{2}-1} = \frac{5}{2}\times 2^{\frac{3}{2}} = 5\sqrt{2} .$$

Example 9.35

Do the limits of following functions exist as $x \to 0$? State reasons for your answer.

(i) $\frac{\sin|x|}{x}$ (ii) $\frac{\sin x}{|x|}$ (iii) $\frac{x\lfloor x\rfloor}{\sin|x|}$ (iv) $\frac{\sin\left(x-\lfloor x\rfloor\right)}{x-\lfloor x\rfloor}$.

Solution

(i)
$$f(x) = \begin{cases} \frac{\sin(-x)}{x} & \text{if } -1 < x < 0 \\ \frac{\sin x}{x} & \text{if } 0 < x < 1 \end{cases}$$

Therefore, $\lim_{x\to 0^-} f(x) = -1$ and

$\lim_{x\to 0^+} f(x) = +1$.

Hence the limit does not exist. Note that $f(0^-) \neq f(0^+)$.

(ii)
$$\frac{\sin x}{|x|} = \begin{cases} \dfrac{\sin(x)}{-x} & \text{if } -1 < x < 0 \\ \dfrac{\sin x}{x} & \text{if } 0 < x < 1 \end{cases}$$

Therefore,
$$\lim_{x \to 0^-} f(x) = -1$$
$$\lim_{x \to 0^+} f(x) = 1$$

Hence the limit does not exist.

(iii)
$$f(x) = \frac{x\lfloor x \rfloor}{\sin|x|} = \begin{cases} \dfrac{-x}{\sin(-x)} & \text{if } -1 < x < 0 \\ \dfrac{x.0}{\sin x} & \text{if } 0 < x < 1 \end{cases}$$
$$= \begin{cases} \dfrac{x}{\sin x} & \text{if } -1 < x < 0 \\ 0 & \text{if } 0 < x < 1 \end{cases}$$

Therefore,
$$\lim_{x \to 0^-} f(x) = +1$$
$$\lim_{x \to 0^+} f(x) = 0.$$

Hence the limit does not exist.

(iv)
$$\frac{\sin\left(x - \lfloor x \rfloor\right)}{x - \lfloor x \rfloor} = \begin{cases} \dfrac{\sin(x-(-1))}{x-(-1)} & \text{if } -1 < x < 0 \\ \dfrac{\sin(x-0)}{x-0} & \text{if } 0 < x < 1 \end{cases}$$
$$f(x) = \begin{cases} \dfrac{\sin(x+1)}{(x+1)} & \text{if } -1 < x < 0 \\ \dfrac{\sin x}{x} & \text{if } 0 < x < 1 \end{cases}$$
$$\lim_{x \to 0^-} f(x) = \frac{\sin 1}{1} = \sin 1$$
$$\lim_{x \to 0^+} f(x) = 1.$$

Hence the limit does not exist.

EXERCISE 9.4

Evaluate the following limits :

(1) $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{7x}$

(2) $\lim\limits_{x \to 0} (1+x)^{1/3x}$

(3) $\lim\limits_{x \to \infty} \left(1 + \dfrac{k}{x}\right)^{\frac{m}{x}}$

(4) $\lim\limits_{x \to \infty} \left(\dfrac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$

(5) $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^{x+2}$

(6) $\lim\limits_{x\to 0}\frac{\sin^3\left(\frac{x}{2}\right)}{x^3}$

(7) $\lim\limits_{x\to 0}\frac{\sin\alpha x}{\sin\beta x}$

(8) $\lim\limits_{x\to 0}\frac{\tan 2x}{\sin 5x}$

(9) $\lim\limits_{\alpha\to 0}\frac{\sin(\alpha^n)}{(\sin\alpha)^m}$

(10) $\lim\limits_{x\to 0}\frac{\sin(a+x)-\sin(a-x)}{x}$

(11) $\lim\limits_{x\to 0}\frac{\sqrt{x^2+a^2}-a}{\sqrt{x^2+b^2}-b}$

(12) $\lim\limits_{x\to 0}\frac{2\arcsin x}{3x}$

(13) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$

(14) $\lim\limits_{x\to 0}\frac{\tan 2x}{x}$

(15) $\lim\limits_{x\to 0}\frac{2^x-3^x}{x}$

(16) $\lim\limits_{x\to 0}\frac{3^x-1}{\sqrt{x+1}-1}$

(17) $\lim\limits_{x\to 0}\frac{1-\cos^2 x}{x\sin 2x}$

(18) $\lim\limits_{x\to\infty}x\left[3^{\frac{1}{x}}+1-\cos\left(\frac{1}{x}\right)-e^{\frac{1}{x}}\right]$

(19) $\lim\limits_{x\to\infty}\{x[\log(x+a)-\log(x)]\}$

(20) $\lim\limits_{x\to\pi}\frac{\sin 3x}{\sin 2x}$

(21) $\lim\limits_{x\to\pi}(1+\sin x)^{2\operatorname{cosec} x}$

(22) $\lim\limits_{x\to 0}\frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

(23) $\lim\limits_{x\to 0}\frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\tan x}$

(24) $\lim\limits_{x\to\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

(25) $\lim\limits_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$

(26) $\lim\limits_{x\to 0}\frac{e^{ax}-e^{bx}}{x}$

(27) $\lim\limits_{x\to 0}\frac{\sin x(1-\cos x)}{x^3}$

(28) $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$

9.3 Continuity

One of the chief features in the behaviour of functions is the property known as continuity. It reflects mathematically the general trait of many phenomena observed by us in nature. For instance, we speak of the continuous expansion of a rod on heating, of the continuous growth of an organism, of a continuous flow, or a continuous variation of atmospheric temperature etc.

The idea of continuity of a function stems from the geometric notion of "no breaks in a graph". In fact, the name itself derives from the Latin continuere, "to hang together". Nevertheless, to identify continuity with "no breaks in a graph" or "a hanging together" has serious drawbacks, at least from the point of view of applying the concept to the analysis of functions. Accordingly, a premature use of the graph to gain insight into the meaning of continuity is advised against as gravely misleading. However, we will realise later that for functions with interval domains, continuity means essentially that the graph may be traced without lifting the point of the pencil.

The proper and effective way of attitude which allows us to put the concept to work is to correlate continuity with limit. Loosely speaking, to possess the property of continuity will mean "to have a favourable limit". In order to formulate the concept of continuity in terms of limit we must focus our attention at a point. Both continuity and limit are primarily concepts defined at a point, but continuity acquires a global character in a pointwise way.

For motivation, consider the following physical situation. A thermometer T measures temperatures along a given hot wire L.

To each point x on the hot wire L is assigned a temperature readings $t(x)$ on thermometer, T. Suppose, to fix ideas, the temperature recordings are observed to be the same, $250°F$, as one moves along the wire L until a point x_0 is reached on L.

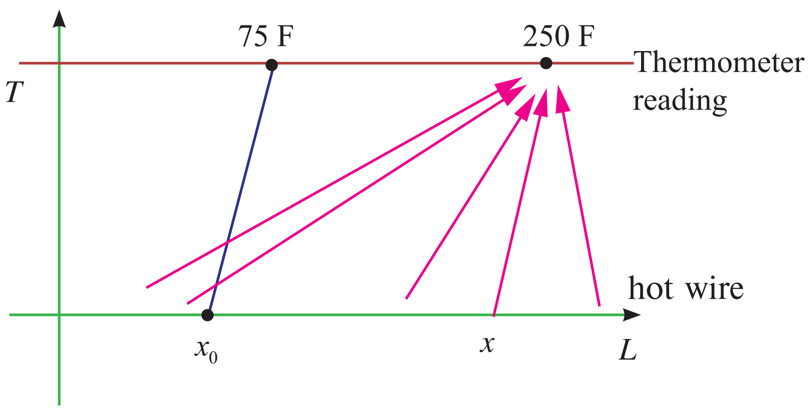


Fig. 9.32

Then suppose that at x_0 the temperature drops suddenly to near room temperature, say $75°F$, as if an insulation were at x_0. But, beyond x_0 suppose readings of $250°F$ are again observed. In function notation we are assuming that

$$t(x) = \begin{cases} 250^{\circ} F & \text{if } x \neq x_0 \\ 75^{\circ} F & \text{if } x = x_0 \end{cases}$$

Thus the point x_0 stands out as singular point ("singular" means "special" or "unusual"). Analyzing the range of temperature readings, we should say that the approach of x to x_0 had no bearing on the approach of the corresponding $t(x)$ values to $t(x_0)$. Briefly, a jump occurs at x_0. Thus we would be led to say that the temperature function "lacks continuity at x_0". For, we would have expected $t(x_0)$ to be $250°F$ since the x values neighbouring x_0 showed $t(x) = 250°F$. We now abstract the notion of continuity, and demand that images be "close" when pre-images are "close". In our example the points on the hot wire were the pre-images, while the temperature readings there were the corresponding images.

The students should reflect on the intuitive idea of continuity by considering instead the contrasting idea of lack of continuity, or more simply "discontinuity" as manifested in every day experiences of abrupt changes which could be headed "then suddenly!". A few that come readily to mind are listed below along with functions which correspond as mathematical models:

(1) Switching on a light : light intensity as a function of time.

(2) Collision of a vehicle : Velocity as a function of time.

(3) Switching off a radio : Sound intensity as a function of time.

(4) Busting of a balloon: Radius as a function of air input.

(5) Breaking of a string : Tension as a function of length.

(6) Cost of postage : Postage as a function of weight.

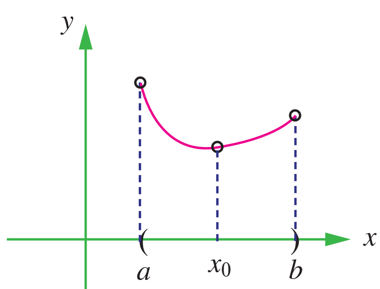
(7) Income tax : Tax-rate as a function of taxable income.

(8) Age count in years : Age in whole years as a function of time.

(9) Cost of insurance premium : Premium as a function of age.

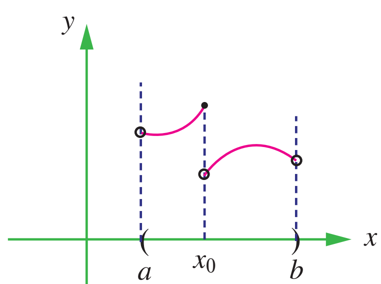
Actually, examples (1) – (5) are not quite accurate. For example, light intensity has a transparent but continuous passage from zero luminosity to positive luminosity. Indeed, nature appears to abhor discontinuity. On the other hand, (6) – (9) are in fact discontinuous and really show jumps at certain points.

Students are advised to relate the mathematical definition of continuity corresponds closely with the meaning of the word continuity in everyday language. A continuous process is one that takes place gradually, without interruption or abrupt change. That is, there are no holes, jumps or gaps. Following figure identifies three values of x at which the graph of a function f is not continuous. At all other points in the interval (a, b), the graph of f is uninterrupted and continuous.



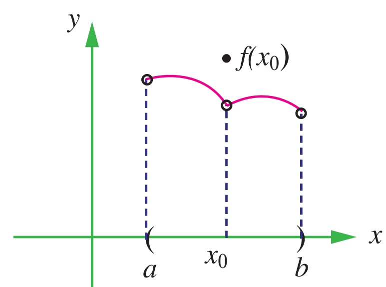
$f(x_0)$ is not defined

Fig. 9.33



both $f(x_0^-)$ and $f(x_0^+)$ exist, but not equal

Fig. 9.34



$\lim_{x \to x_0} f(x) \neq f(x_0)$

Fig. 9.35

Looking at the above graphs (9.33 to 9.35), three conditions exist for which the graph of f is not continuous at $x = x_0$.

It appears that continuity at $x = x_0$ can be destroyed by any one of the following three conditions:

(1) The function is not defined at $x = x_0$.

(2) The limit of $f(x)$ does not exist at $x = x_0$.

(3) The limit of $f(x)$ exists at $x = x_0$, but, it is not equal to $f(x_0)$.

Now let us look at the illustrative examples

Illustration 9.6

(i) $f(x) = x^2+3$ (ii) $f(x) = \dfrac{16-x^2}{4+x}$

(i) As $x \to 2$, the one sided limits are

$$\lim_{x \to 2^-} f(x) = 7$$

$$\lim_{x \to 2^+} f(x) = 7$$

and hence $\lim_{x \to 2} f(x) = 7$ and moreover $f(2)$ is defined and $f(2) = 7 = \lim_{x \to 2} f(x)$. In this case $f(x)$ is continuous at $x = 2$.

(ii) The one sided limits are : $\lim_{x \to -4^-} f(x) = 8$

$$\lim_{x \to -4^+} f(x) = 8$$

and therefore $\lim_{x \to -4} f(x) = 8$ but $f(-4)$ does not exist.

Note that although $\lim_{x\to -4} f(x)$ exists, the function value at -4, namely $f(-4)$ is not defined. Thus the existence of $\lim_{x\to -4} f(x)$ has no bearing on the existence of $f(-4)$.

Now we formally define continuity as in

Definition 9.7

Let I be an open interval in $\mathbb{R}$ containing x_0. Let $f : I \to \mathbb{R}$. Then f is said to be continuous at x_0 if it is defined in a neighbourhood of this point and if the limit of this function, as the independent variable x tends to x_0, exists and is equal to the value of the function at $x = x_0$.

Thus three requirements have to be satisfied for the continuity of a function $y = f(x)$ at $x = x_0$:

(i) $f(x)$ must be defined in a neighbourhood of x_0 (i.e., $f(x_0)$ exists);

(ii) $\lim_{x\to x_0} f(x)$ exists ;

(iii) $f(x_0) = \lim_{x\to x_0} f(x)$.

The condition (iii) can be reformulated as $\lim_{\Delta x\to 0}\left[f(x_0+\Delta x) - f(x_0)\right] = 0$ and the continuity of f at x_0 can be restated as follows :

Definition 9.8

A function $y = f(x)$ is said to be continuous at a point x_0 (or at $x = x_0$) if it is defined in some neighbourhood of x_0 and if $\lim_{\Delta x\to 0}\left[f(x_0+\Delta x) - f(x_0)\right] = 0$.

The condition (iii) can also be put in the form $\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right)$. Thus, if the symbol of the limit and the symbol of the function can be interchanged, the function is continuous at the limiting value of the argument.

9.3.1 Examples of functions Continuous at a point

(1) Constant function is continuous at each point of $\mathbb{R}$.

Let $f(x) = k, k \in \mathbb{R}$ is constant. If $x_0 \in \mathbb{R}$, then $f(x_0) = k$.

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0}(k) = k.$$

(2) Power functions with positive integer exponents are continuous at every point of $\mathbb{R}$

If $f(x) = x^n$, domain of f is $\mathbb{R} = (-\infty, \infty)$ and $\lim_{x\to x_0} x^n = x_o^n, x_0 \in \mathbb{R}$ by the limit theorem.

(3) Polynomial functions, $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n, \quad a_0 \neq 0$

are continuous at every point of $\mathbb{R}$. By limit theorem,

$$\lim_{x\to x_0} p(x) = a_0x_0^n + a_1x_0^{n-1} + \cdots + a_{n-1}x_0 + a_n = p(x_0).$$

(4) Quotients of polynomials namely rational functions of the form

$R(x) = \dfrac{p(x)}{q(x)}$, are continuous at every point where $q(x) \neq 0$, and

$$\lim_{x \to x_0} R(x) = \lim_{x \to x_0} \frac{p(x)}{q(x)}$$

$$= \frac{\lim_{x \to x_0} p(x)}{\lim_{x \to x_0} q(x)} = \frac{p(x_0)}{q(x_0)} = R(x_0).$$

(5) The circular functions $\sin x$ and $\cos x$ are continuous at every point of their domain $\mathbb{R} = (-\infty, \infty)$ since $\lim_{x \to x_0} \sin x = \sin x_0$, $\lim_{x \to x_0} \cos x = \cos x_0$.

As a consequence, tan x, cot x, cosec x, sec x are continuous on their proper domains in view of the reciprocal and quotient rules in the algebra of limits.

(6) The nth root functions, $f(x) = x^{\frac{1}{n}}$ are continuous in their proper domain since $\lim_{x \to x_0} \left(x^{\frac{1}{n}} \right) = x_0^{\frac{1}{n}}$.

(7) The reciprocal function $f(x) = \dfrac{1}{x}$ is not defined at 0 and hence it is not continuous at 0. It is continuous at each point of $\mathbb{R} - \{0\}$.

(8) $h(x) = \begin{cases} x+1, & x \leq 0 \\ x^2+1, & x > 0 \end{cases}$

The domain of h is all of real numbers and

$$\lim_{x \to 0^-} h(x) = \lim_{x \to 0^-} (x+1) = 1 = h(0)$$

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} (x^2+1) = 1 = h(0).$$

Thus $h(x)$ is continuous at $x = 0$.

Indeed, $h(x)$ is continuous at each point of $(-\infty, 0)$ and each point of $(0, \infty)$ and hence h is continuous in the whole of $(-\infty, \infty)$.

(9) The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0$.

For, $\lim_{x \to 0^-} \lfloor x \rfloor = -1$ and

$$\lim_{x \to 0^+} \lfloor x \rfloor = 0$$

It is discontinuous at each integer point. In fact,

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \text{ and}$$

$$\lim_{x \to n^+} \lfloor x \rfloor = n.$$

(10) The modulus function

$$f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

is continuous at all points of the real line $\mathbb{R}$.

In particular,

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0,$$

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} (x) = 0, \text{ and}$$

$$\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x) = 0 = f(0).$$

(11) The exponential function $f(x) = e^x$ is continuous on $\mathbb{R}$.

(12) The logarithmic function $f(x) = \log x$ $(x > 0)$ in continuous in $(0, \infty)$

9.3.2 Algebra of continuous functions

If f and g are continuous at x_0 then

(1) $f + g$ is continuous at $x = x_0$,

(2) $f - g$ is continuous at $x = x_0$,

(3) $f \,.\, g$ is continuous at $x = x_0$, and

(4) $\dfrac{f}{g}$ is continuous at $x = x_0$ $(g(x) \neq 0)$.

(5) Composite function theorem on continuity.

If f is continuous at $g(x_0)$ and g is continuous at x_0 then fog is continuous at x_0.

Continuity in a closed interval

Definition 9.9

A function $f : [a, b] \to \mathbb{R}$ is said to be continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \text{ and } \lim_{x \to b^-} f(x) = f(b).$$

That is, the function f is continuous from the right at a and continuous from the left at b, and is continuous at each point $x_0 \in (a, b)$.

Illustration 9.7

Discuss the continuity of $f(x) = \sqrt{1 - x^2}$.

The domain of definition of f is the closed interval $[-1, 1]$.

(f is defined if $1 - x^2 \geq 0$)

For any point $c \in (-1, 1)$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\sqrt{1-x^2} = \left[\lim_{x\to c}(1-x^2)\right]^{\frac{1}{2}}$$

$$= (1-c^2)^{\frac{1}{2}} = f(c).$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+}(1-x^2)^{\frac{1}{2}} = 0 = f(1).$$

$$\lim_{x\to -1^-} f(x) = \left[\lim_{x\to -1^-}(1-x^2)\right]^{\frac{1}{2}} = 0 = f(-1).$$

Thus f is continuous on $[-1, 1]$. One can also solve this problem using composite function theorem.

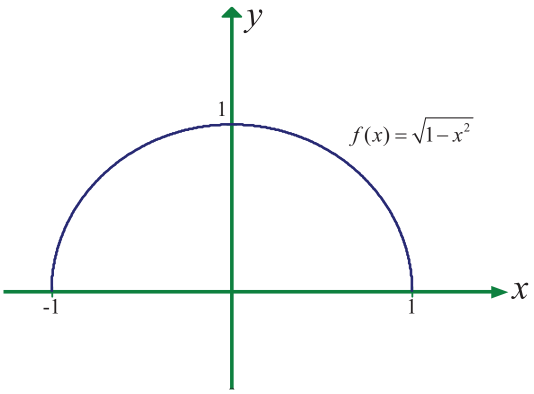


Fig. 9.36

Example 9.36

Describe the interval(s) on which each function is continuous.

(i) $f(x) = \tan x$

(ii) $g(x) = \begin{cases} \sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(iii) $h(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution

(i) The tangent function $f(x) = \tan x$ is undefined at $x = (2n+1)\dfrac{\pi}{2}, n \in Z$.

At all other points it is continuous, so $f(x) = \tan x$ is continuous on each of the open intervals

$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots$$

(ii) The function $y = \dfrac{1}{x}$ is continuous at all points of $\mathbb{R}$ except at $x = 0$ where it is undefined.

The function $g(x) = \sin\dfrac{1}{x}$ is continuous at all points except $x = 0$, where $\lim\limits_{x\to 0} g(x)$ does not exist. So, g is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$

(iii) The function $h(x)$ is defined at all points of the real line $\mathbb{R} = (-\infty, \infty)$; for any $x_0 \neq 0$,

$$\lim_{x\to x_0} h(x) = \left(\lim_{x\to x_0} x\sin\frac{1}{x}\right)$$

$$= x_0 \sin\frac{1}{x_0} = h(x_0)$$

For $x_0 = 0$

$$h(x) = x.\sin\frac{1}{x}$$

$$-x \leq x\sin\frac{1}{x} \leq x$$

$$g(x) = -x, f(x) = x\sin\frac{1}{x}, h(x) = x$$

$$\lim_{x\to 0} g(x) = 0, \lim_{x\to 0} h(x) = 0$$

and have $\lim\limits_{x\to 0} x\sin\frac{1}{x} = 0.$

By Sandwich theorem

$$\lim_{x\to 0}\left(x\sin\frac{1}{x}\right) = 0 = h(0).$$

Therefore $h(x)$ is continuous in the entire real line.

Example 9.37

A tomato wholesaler finds that the price of a newly harvested tomatoes is ₹0.16 per kg if he purchases fewer than 100 kgs each day. However, if he purchases at least 100 kgs daily, the price drops to ₹0.14 per kg. Find the total cost function and discuss the cost when the purchase is 100 kgs.

Solution

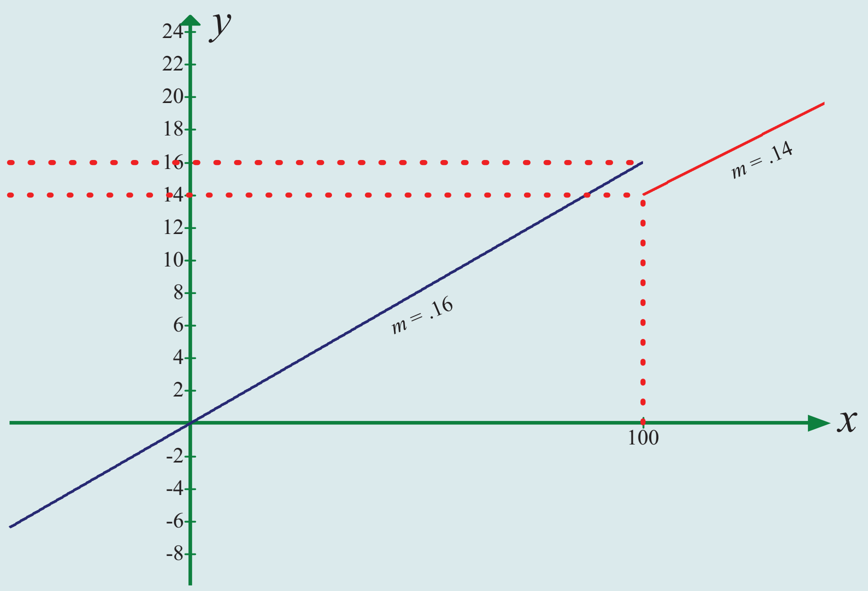


Fig.9.37

Let x denote the number of kilograms bought per day and C denote the cost. Then,

$$C(x) = \begin{cases} 0.16x, & \text{if } 0 \le x < 100 \\ 0.14x, & \text{if } x \ge 100 \end{cases}.$$

The sketch of this function is shown in Fig. 9.37.

It is discontinuous at $x = 100$ since $\lim\limits_{x\to 100^-} C(x) = 16$ and $\lim\limits_{x\to 100^+} C(x) = 14$.

Note that $C(100) = 14$. Thus, $\lim\limits_{x\to 100^-} C(x) = 16 \ne 14 = \lim\limits_{x\to 100^+} C(x) = C(100)$.

Note also that the function jumps from one finite value 14 to another finite value 16.

9.3.3 Removable and Jump Discontinuities

Let us look at the following functions :

(i) $f(x) = \dfrac{\sin x}{x}$

(ii) $g(x) = C(x)$, where $C(x)$ is as defined in Example 9.38.

The function $f(x)$ is defined at all points of the real line except $x = 0$. That is, $f(0)$ is undefined, but $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ exists. If we redefine the function $f(x)$ as

$$h(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

h is defined at all points of the real line including $x = 0$. Moreover, h is continuous at $x = 0$ since

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1 = h(0).$$

Note that $h(x) = f(x)$ for all $x \neq 0$. Even though the original function $f(x)$ fails to be continuous at $x = 0$, the redefined function became continuous at 0. That is, we could remove the discontinuity by redefining the function. Such discontinuous points are called removable discontinuities. This example leads us to have the following.

Definition 9.10

A function f defined on an interval $I \subseteq \mathbb{R}$ is said to have removable discontinuity at $x_0 \in I$ if there is a function $h : I \to \mathbb{R}$ such that $h(x) = \begin{cases} f(x), & \text{if } x \neq x_0 \\ \lim\limits_{x \to x_0} f(x), & \text{if } x = x_0 \end{cases}$.

Note that for removable discontinuity, $\lim\limits_{x \to x_0} f(x)$ must exist.

Now if we examine the function $g(x) = C(x)$ (see Example 9.38), eventhough it is defined at all points of $[0, \infty)$, $\lim\limits_{x \to 100} g(x)$ does not exist and it has a jump of height $\lim\limits_{x \to 100^+} g(x) - \lim\limits_{x \to 100^-} g(x) = 16 - 14 = 2$, which is finite. Since $\lim\limits_{x \to 100} g(x)$ does not exist, it is not continuous at $x = 100$. Such discontinuities are called jump discontinuities. Thus we have the following :

Definition 9.11

Let f be a function defined on an interval $I \subseteq \mathbb{R}$. Then f is said to have jump discontinuity at a point $x_0 \in I$ if f is defined at x_0,

$$\lim_{x \to x_0^-} f(x) \text{ and } \lim_{x \to x_0^+} f(x) \text{ exist but}$$

$$\lim_{x \to x_0^-} f(x) \neq \lim_{x \to x_0^+} f(x).$$

Example 9.38

Determine if f defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is continuous in $\mathbb{R}$.

Solution

By Sandwich theorem $\lim\limits_{x\to 0} x^2 \sin\frac{1}{x} = 0$ and $f(0) = 0$ by the definition of $f(x)$. Hence it is continuous at $x = 0$. For other values it is clearly continuous and hence continuous in $\mathbb{R}$.

EXERCISE 9.5

(1) Prove that $f(x) = 2x^2 + 3x - 5$ is continuous at all points in $\mathbb{R}$.

(2) Examine the continuity of the following :

(i) $x + \sin x$ (ii) $x^2 \cos x$ (iii) $e^x \tan x$

(iv) $e^{2x} + x^2$ (v) $x . \ln x$ (vi) $\frac{\sin x}{x^2}$

(vii) $\frac{x^2 - 16}{x + 4}$ (viii) $|x + 2| + |x - 1|$ (ix) $\frac{|x-2|}{|x+1|}$

(x) $\cot x + \tan x$

(3) Find the points of discontinuity of the function f, where

(i) $f(x) = \begin{cases} 4x+5, & \text{if } x \leq 3 \\ 4x-5, & \text{if } x > 3 \end{cases}$ (ii) $f(x) = \begin{cases} x+2, & \text{if } x \geq 2 \\ x^2, & \text{if } x < 2 \end{cases}$

(iii) $f(x) = \begin{cases} x^3 - 3, & \text{if } x \leq 2 \\ x^2 + 1, & \text{if } x > 2 \end{cases}$ (iv) $f(x) = \begin{cases} \sin x, & 0 \leq x \leq \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$

(4) At the given point x_0 discover whether the given function is continuous or discontinuous citing the reasons for your answer :

(i) $x_0 = 1,\ f(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ (ii) $x_0 = 3,\ f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{if } x \neq 3 \\ 5, & \text{if } x = 3 \end{cases}$

(5) Show that the function $\begin{cases} \frac{x^3-1}{x-1}, & \text{if } x \neq 1 \\ 3, & \text{if } x = 1 \end{cases}$ is continuous on $(-\infty, \infty)$

(6) For what value of α is this function $f(x) = \begin{cases} \frac{x^4-1}{x-1}, & \text{if } x \neq 1 \\ \alpha, & \text{if } x = 1 \end{cases}$ continuous at $x = 1$?

(7) Let $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ x^2, & \text{if } 0 \le x < 2 \\ 4, & \text{if } x \ge 2 \end{cases}$ .Graph the function. Show that $f(x)$ continuous on $(-\infty, \infty)$.

(8) If f and g are continuous functions with $f(3) = 5$ and $\lim\limits_{x \to 3}\left[2f(x) - g(x)\right] = 4$, find $g(3)$.

(9) Find the points at which f is discontinuous. At which of these points f is continuous from the right, from the left, or neither? Sketch the graph of f.

(i) $f(x) = \begin{cases} 2x+1, & \text{if } x \le -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x-1, & \text{if } x \ge 1 \end{cases}$ (ii) $f(x) = \begin{cases} (x-1)^3, & \text{if } x < 0 \\ (x+1)^3, & \text{if } x \ge 0 \end{cases}$

(10) A function f is defined as follows :

$$f(x) = \begin{cases} 0 & \text{for } x < 0; \\ x & \text{for } 0 \le x < 1; \\ -x^2 + 4x - 2 & \text{for } 1 \le x < 3; \\ 4 - x & \text{for } x \ge 3 \end{cases}$$

Is the function continuous?

(11) Which of the following functions f has a removable discontinuity at $x = x_0$? If the discontinuity is removable, find a function g that agrees with f for $x \ne x_0$ and is continuous on $\mathbb{R}$.

(i) $f(x) = \dfrac{x^2 - 2x - 8}{x+2}$, $x_0 = -2$.

(ii) $f(x) = \dfrac{x^3 + 64}{x+4}$, $x_0 = -4$.

(iii) $f(x) = \dfrac{3 - \sqrt{x}}{9 - x}$, $x_0 = 9$.

(12) Find the constant b that makes g continuous on $(-\infty, \infty)$.

$$g(x) = \begin{cases} x^2 - b^2 & \text{if } x < 4 \\ bx + 20 & \text{if } x \ge 4 \end{cases}$$

(13) Consider the function $f(x) = x \sin \dfrac{\pi}{x}$. What value must we give $f(0)$ in order to make the function continuous everywhere?

(14) The function $f(x) = \dfrac{x^2 - 1}{x^3 - 1}$ is not defined at $x = 1$. What value must we give $f(1)$ inorder to make $f(x)$ continuous at $x = 1$?

(15) State how continuity is destroyed at $x = x_0$ for each of the following graphs.

(a)

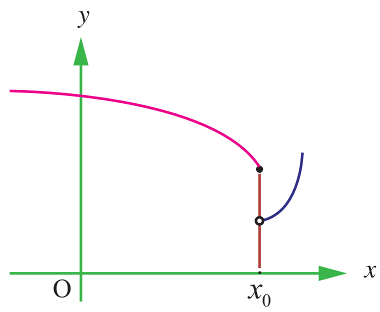


Fig. 9.38

(b)

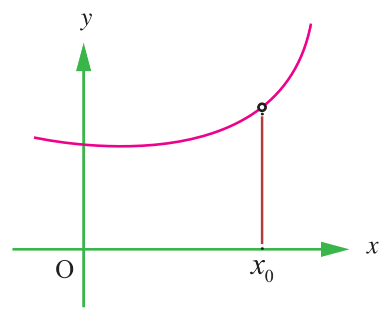


Fig. 9.39

(c)

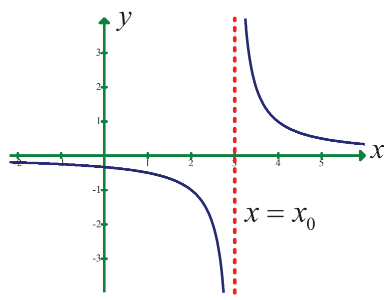


Fig. 9.40

(d)

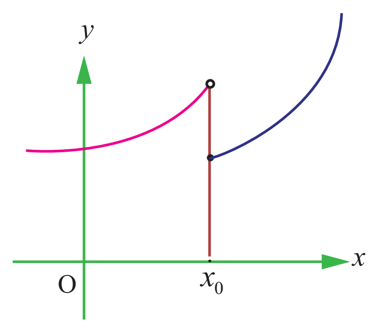


Fig. 9.41

EXERCISE 9.6

Choose the correct or the most suitable answer from the given four alternatives.

(1) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

(1) 1 (2) 0 (3) ∞ (4) $-\infty$

(2) $\lim\limits_{x\to\pi/2} \dfrac{2x-\pi}{\cos x}$

(1) 2 (2) 1 (3) -2 (4) 0

BC1WLY

(3) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-\cos 2x}}{x}$

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) does not exist

(4) $\lim\limits_{\theta\to 0} \dfrac{\sin\sqrt{\theta}}{\sqrt{\sin\theta}}$

(1) 1 (2) -1 (3) 0 (4) 2

(5) $\lim\limits_{x\to\infty} \left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x$ is

(1) e^4 (2) e^2 (3) e^3 (4) 1

(6) $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2-1}}{2x+1}=$

(1) 1 (2) 0 (3) -1 (4) $\dfrac{1}{2}$

(7) $\lim\limits_{x\to 0}\dfrac{a^x-b^x}{x}=$

(1) $\log ab$ (2) $\log\left(\dfrac{a}{b}\right)$ (3) $\log\left(\dfrac{b}{a}\right)$ (4) $\dfrac{a}{b}$

(8) $\lim\limits_{x\to 0}\dfrac{8^x-4^x-2^x+1^x}{x^2}=$

(1) 2 log 2 (2) $2(\log 2)^2$ (3) log 2 (4) 3 log 2

(9) If $f(x)=x(-1)^{\left\lfloor \frac{1}{x}\right\rfloor}$, $x\le 0$, then the value of $\lim\limits_{x\to 0}f(x)$ is equal to

(1) -1 (2) 0 (3) 2 (4) 4

(10) $\lim\limits_{x\to 3}\lfloor x\rfloor=$

(1) 2 (2) 3 (3) does not exist (4) 0

(11) Let the function f be defined by $f(x)=\begin{cases}3x & 0\le x\le 1\\ -3x+5 & 1<x\le 2\end{cases}$, then

(1) $\lim\limits_{x\to 1}f(x)=1$ (2) $\lim\limits_{x\to 1}f(x)=3$

(3) $\lim\limits_{x\to 1}f(x)=2$ (4) $\lim\limits_{x\to 1}f(x)$ does not exist

(12) If $f:\mathbb{R}\to\mathbb{R}$ is defined by $f(x)=\lfloor x-3\rfloor+|x-4|$ for $x\in\mathbb{R}$, then $\lim\limits_{x\to 3^-}f(x)$ is equal to

(1) -2 (2) -1 (3) 0 (4) 1

(13) $\lim\limits_{x\to 0}\dfrac{xe^x-\sin x}{x}$ is

(1) 1 (2) 2 (3) 3 (4) 0

(14) If $\lim\limits_{x\to 0}\dfrac{\sin px}{\tan 3x}=4$, then the value of p is

(1) 6 (2) 9 (3) 12 (4) 4

(15) $\lim\limits_{\alpha\to\pi/4}\dfrac{\sin\alpha-\cos\alpha}{\alpha-\dfrac{\pi}{4}}$ is

(1) $\sqrt{2}$ (2) $\dfrac{1}{\sqrt{2}}$ (3) 1 (4) 2

(16) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dfrac{3}{n^2}+\dots+\dfrac{n}{n^2}\right)$ is

(1) $\dfrac{1}{2}$ (2) 0 (3) 1 (4) ∞

(17) $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1}{x}=$

(1) 1 (2) e (3) $\dfrac{1}{e}$ (4) 0

(18) $\lim\limits_{x\to 0}\dfrac{e^{\tan x}-e^{x}}{\tan x - x}=$

(1) 1 (2) e (3) $\dfrac{1}{2}$ (4) 0

(19) The value of $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt{x^2}}$ is

(1) 1 (2) – 1 (3) 0 (4) limit does not exist

(20) The value of $\lim\limits_{x\to k^-} x-\lfloor x \rfloor$, where k is an integer is

(1) – 1 (2) 1 (3) 0 (4) 2

(21) At $x=\dfrac{3}{2}$ the function $f(x)=\dfrac{|2x-3|}{2x-3}$ is

(1) continuous (2) discontinuous (3) differentiable (4) non-zero

(22) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x & x \text{ is irrational} \\ 1-x & x \text{ is rational}\end{cases}$ then f is

(1) discontinuous at $x=\dfrac{1}{2}$ (2) continuous at $x=\dfrac{1}{2}$

(3) continuous everywhere (4) discontinuous everywhere

(23) The function $f(x)=\begin{cases} \dfrac{x^2-1}{x^3+1} & x\neq -1 \\ P & x=-1\end{cases}$ is not defined for $x=-1$. The value of $f(-1)$ so that the function extended by this value is continuous is

(1) $\dfrac{2}{3}$ (2) $-\dfrac{2}{3}$ (3) 1 (4) 0

(24) Let f be a continuous function on [2, 5]. If f takes only rational values for all x and $f(3)=12$, then $f(4.5)$ is equal to

(1) $\dfrac{f(3)+f(4.5)}{7.5}$ (2) 12 (3) 17.5 (4) $\dfrac{f(4.5)-f(3)}{1.5}$

(25) Let a function f be defined by $f(x)=\dfrac{x-|x|}{x}$ for $x\neq 0$ and $f(0)=2$. Then f is

(1) continuous nowhere (2) continuous everywhere

(3) continuous for all x except $x = 1$ (4) continuous for all x except $x = 0$

SUMMARY

In this chapter we have acquired the knowledge of

- Limit of a function $y = f(x)$ as x approaches x_0 from the lower values of x_0.
- Limit of $y = f(x)$ as x approaches x_0 from the higher values of x_0.
- Limit of a function as x approaches x_0, in the deleted neighbourhood of x_0 exists if and only if $\lim\limits_{x \to x_0^-} f(x) = L = \lim\limits_{x \to x_0^+} f(x) = \lim\limits_{x \to x_0} f(x) = L.$
- $\lim\limits_{x \to x_0} f(x) = L$ also means that $f(x)$ converges to L as x approaches x_0 from either side of x_0 except at $x = x_0$.
- If $\lim\limits_{x \to x_0} f(x)$ and $\lim\limits_{x \to x_0} g(x)$ exist then

  (i) $\lim\limits_{x \to x_0} \left[f(x) \pm g(x) \right] = \lim\limits_{x \to x_0} f(x) \pm \lim\limits_{x \to x_0} g(x)$

  (ii) $\lim\limits_{x \to x_0} \left[f(x).g(x) \right] = \lim\limits_{x \to x_0} f(x). \lim\limits_{x \to x_0} g(x)$

  (iii) $\lim\limits_{x \to x_0} \left[\dfrac{f(x)}{g(x)} \right] = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$ if $g(x) \neq 0$ and $\lim\limits_{x \to x_0} g(x) \neq 0$.
- Limit of $f(x)$ as x approaches x_0 does not exist if either $f(x) \to \pm\infty$ as $x \to x_0^-$ or $f(x) \to \pm\infty$ as $x \to x_0^+$ or $\lim\limits_{x \to x_0^-} f(x) = l_1 \neq l_2 = \lim\limits_{x \to x_0^+} f(x)$
- (M, ∞) is the neighborhood of $+\infty$, $M > 0$

  $(-\infty, K)$ is the neighborhood of $-\infty$, $K < 0$.
- If $f(x) \to \pm\infty$ as $x \to x_0$ then $x = x_0$ is a vertical asymptote.
- The line $y = l_1$ (or l_2) is a horizontal asymptote of the curve $y = f(x)$ if either $f(x) \to l_1$ as $x \to \infty$ or $f(x) \to l_2$ as $x \to -\infty$.
- $f(x)$ is continuous at x_0 if and only if

  (i) $\lim\limits_{x \to x_0} f(x) = f(x_0)$

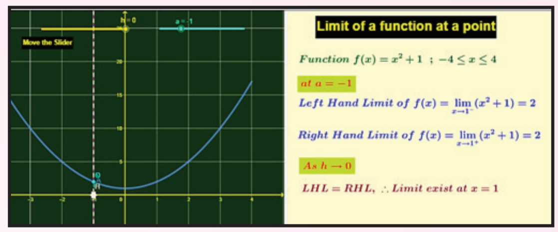
  (ii) $\lim\limits_{\Delta x \to 0} [f(x_0 + \Delta x) - f(x_0)] = 0$

  (iii) $\lim\limits_{x \to x_0} f(x) = f\left(\lim\limits_{x \to x_0} x \right)$.
- Jump and removable discontinuities.

ICT CORNER 9(a)

Limits and continuity

Expected Outcome



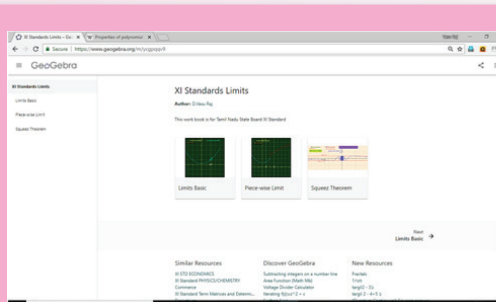
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI Standard Limits" will appear. In that there are several worksheets related to your lesson.

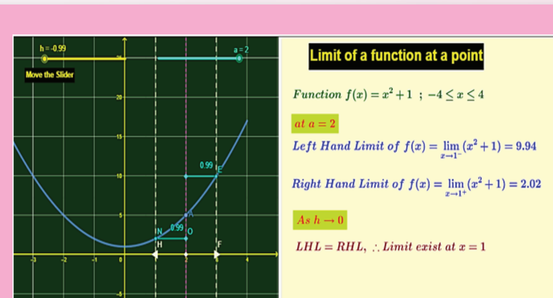
Step 2

Select the work sheet "Limits basic". A continuous function is given. You can select the limit at a desired point by moving the slider "a". Then move the lines $x=h$(nearest to a point) both left and right side to check $f(h)$ by moving the slider "h"

Compare this with the definition given in book.



Step1



Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

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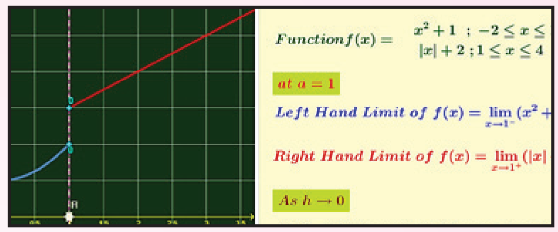
QB365 - Question Bank Software

ICT CORNER 9(b)

Limits and continuity

Expected Outcome



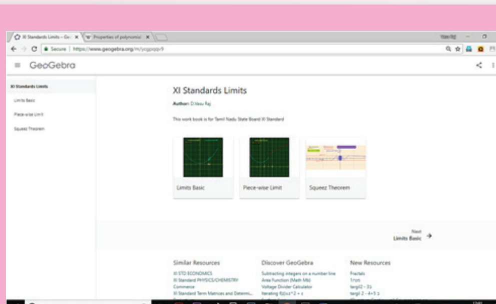
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI Standard Limits" will appear. In that there are several worksheets related to your lesson.

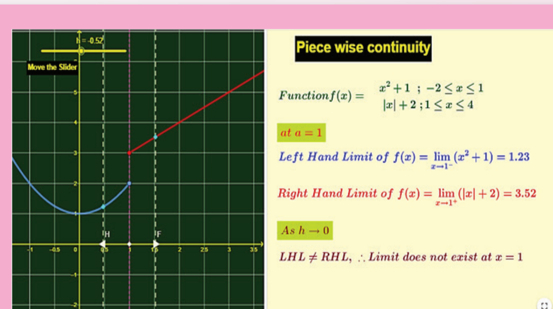
Step 2

Select the work sheet "Piece-wise limit". Piece-wise function is given. Move the lines $x=h$(nearest to $x = 1$) both left and right side to check $f(h)$ by moving the slider "h"

Compare this with the definition given in book.



Step1



Step2

Browse in the link:

Matrices and Determinants: https://ggbm.at/cpknpvvh

B162_11_MAT_EM

Chapter 10 Differential Calculus- Differentiability and Methods of Differentiation

BCASNL

"Take what you need,
do what you should, you will get what you want"

- Leibnitz

10.1 Introduction

In this chapter we discuss the concept of derivative and related concepts and develop tools necessary for solving real life problems. In this connection, let us look at the following problem of finding average velocity.

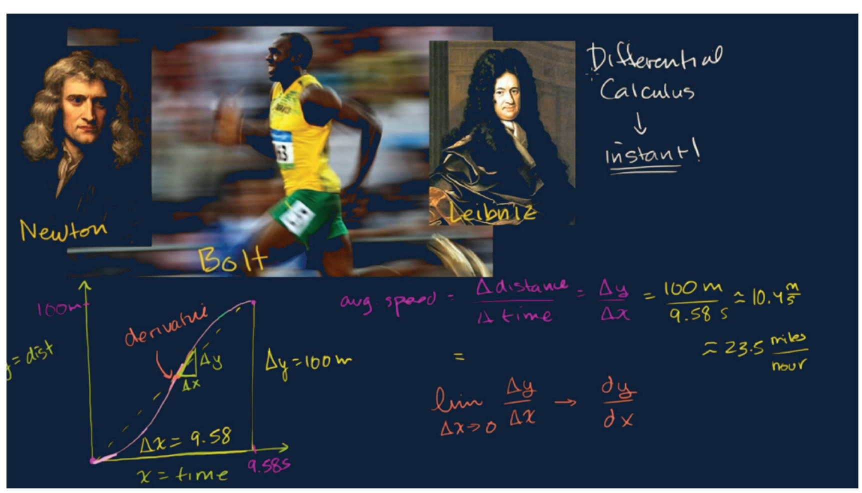
Almost everyone has an intuitive notion of speed or velocity as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 km in one hour, the average velocity of the bus must have been 60 km/h. Of course, it is difficult to maintain this rate of 60 km/h for the entire trip because the bus slows down for towns and speeds up when it passes cars. In other words, the velocity changes with time. If a bus company's schedule demands that the bus travel 60 km from one town to another in one hour, the driver knows instinctively that he must compensate for velocities or speeds greater than this at other points in the journey. Knowing that the average velocity is 60 km/h does not, however, answer the question: What is the velocity of the bus at a particular instant?

In general, this average velocity or average speed of a moving object is the time rate of change of position defined by

$$v_{ave} = \frac{\text{distance travelled}}{\text{time of travel}}$$

Consider a runner who finishes a 10 km race in an elapsed time of 1 h 15 min (1.25 h). The runner's average velocity or average speed for this race is

$$v_{ave} = \frac{10}{1.25} = 8.$$



Usain Bolt's average speed

$$\frac{\Delta y}{\Delta x} = \frac{100m}{9.58s} = 10.4$$

How fast is Usain Bolt right now? → Calculus

But suppose we now wish to determine the runner's exact velocity v at the instant the runner is one-half into the race. If the distance run in the time interval from 0 h to 0.5 h is measured to be 5 km, then $v_{ave} = \frac{5}{0.5} = 10$.

Again this number is not a measure or necessarily such a good indicator, of the instantaneous rate v at which the runner is moving 0.5 h into the race. If we determine that rate at 0.6 h the runner is 5.7 km from the starting line, then the average velocity from 0 h to 0.6 h is

$$v_{ave} = \frac{5.7 - 5}{0.6 - 0.5} = 7km/h$$

Gottfried Wilhelm Leibnitz (1646 - 1716)

The latter number is a more realistic measure of the rate v. By "shrinking" the time interval between 0.5 h and the time that corresponds to a measured position close to 5 km, we expect to obtain even better approximations to the runner's velocity at time 0.5 h.

This problem of finding velocities leads us to deal with the general problem of finding the derivative of a general mathematical model represented by the analytic equation, $y = f(x)$ Consequently, we will move towards in achieving the following objectives and subsequently deal with the analysis of derivatives.

Learning Objectives

On completion of this chapter, the students are expected to

- acquire the concept of a derivative as limit of quotients.
- visualise the concept of derivative geometrically.
- understand derivative as a process of measuring changes.
- realise derivative as a tool to measure slopes of tangents to curves / rates of changes.
- understand different methods of differentiation.
- apply calculus as a tool to solve everyday real life problems.

10.2 The concept of derivative

Calculus grew out of four major problems that mathematicians were working on during the seventeenth century.

(1) The tangent line problem

(2) The velocity and acceleration problem

(3) The minimum and maximum problem

(4) The area problem

We take up the above problems 1 and 2 for discussion in this chapter while the last two problems are dealt with in the later chapters.

10.2.1 The tangent line problem

What does it mean to say that a line is tangent to a curve at a point? For a circle, the tangent line at a point P is the line that is perpendicular to the radial line at a point P, as shown in fig. 10.1.

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For a general curve, however, the problem is more difficult, for example, how would you define the tangent lines shown in the following figures 10.2 to 10.4.

You might say that a line is tangent to a curve at a point P if it touches, but does not cross, the curve at point P. This definition would work for the first curve (Fig. 10.2), but not for the second (Fig. 10.3). Or you might say that a line is tangent to a curve if the line touches or intersects the curve exactly at one point. This definition would work for a circle but not for more general curves, as the third curve shows (Fig. 10.4).

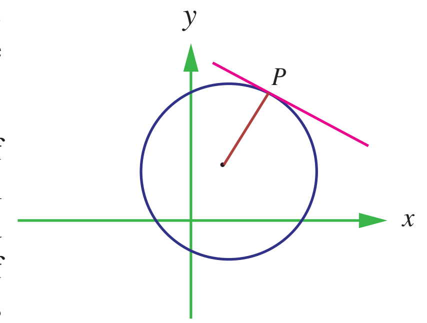


Fig. 10.1

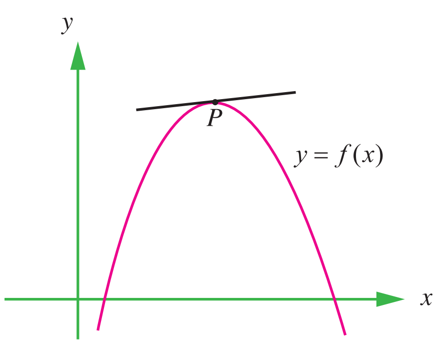


Fig. 10.2

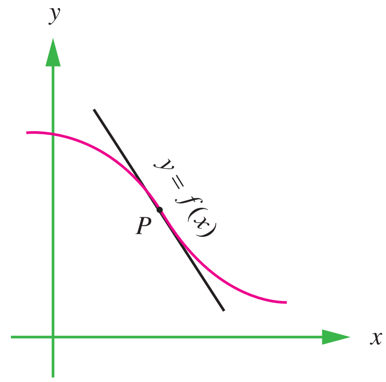


Fig. 10.3

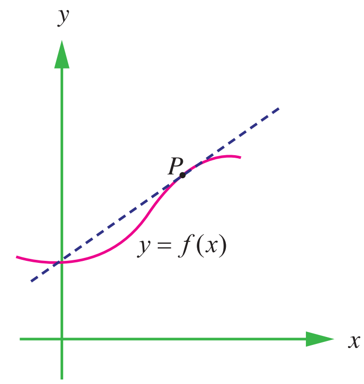


Fig. 10.4

Essentially, the problem of finding the tangent line at a point P boils down to the problem of finding the slope of the tangent line at point P. You can approximate this slope using a secant line through the point of tangency and a second point on the curve as in the following Fig. 10.5.

Let $P(x_0, f(x_0))$ be the point of tangency and $Q(x_0 + \Delta x, f(x_0 + \Delta x))$ be the second point.

The slope of the secant line through the two points is given by substitution into the slope formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{sec} = \frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

That is, $m_{\text{sec}} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$, which is the slope of the secant line.

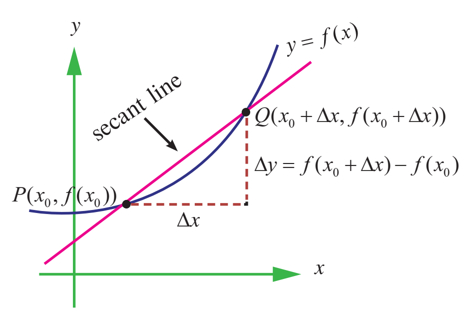


Fig. 10.5

The right hand side of this equation is a **difference quotient.** The denominator Δx is the **change in x** (increment in x), and the numerator $\Delta y = f(x_0 + \Delta x) - f(x_0)$ is the **change in y**.

The beauty of this procedure is that you can obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency.

Tangent line approximation

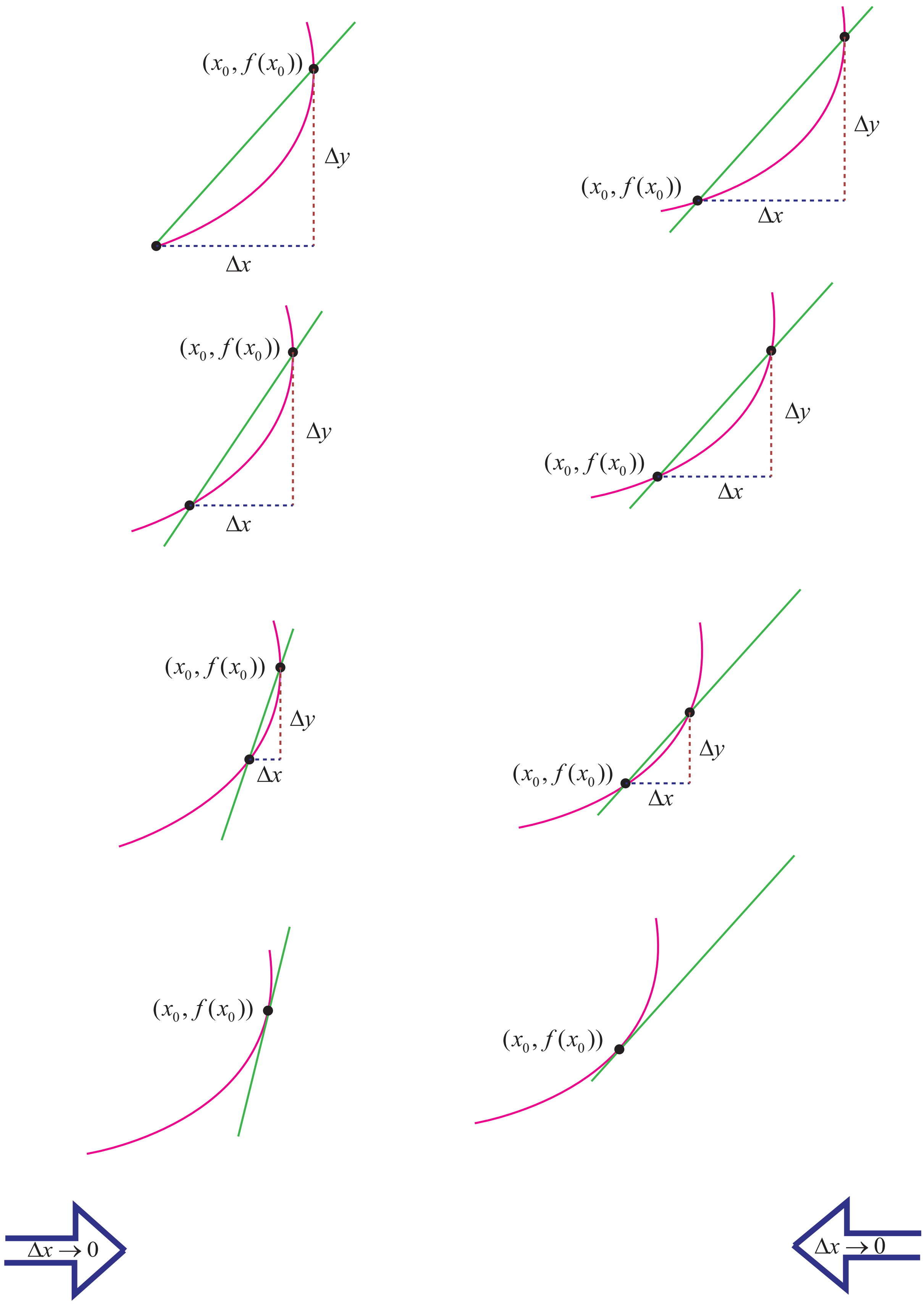


Fig. 10.6 to 10.13

QB365 - Question Bank Software

Illustration 10.1

Let us make an attempt to find the slope of the tangent line to the graph of $f(x) = x^2$ at (1, 1).

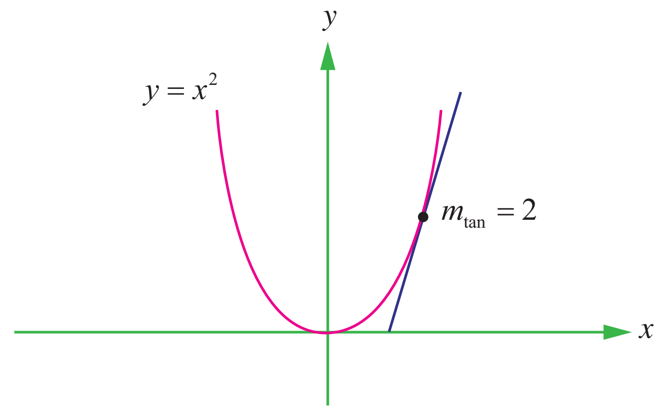


Fig. 10.14

As a start, let us take $\Delta x = 0.1$ and find the slope of the secant line through $(1,1)$ and $(1.1, (1.1)^2)$.

(i) $\quad f(1.1) = (1.1)^2 = 1.21$

(ii) $\quad \Delta y = f(1.1) - f(1)$

$\qquad = 1.21 - 1 = 0.21$

(iii) $\quad \dfrac{\Delta y}{\Delta x} = \dfrac{0.21}{0.1} = 2.1$

Tabulate the successive values to the right and left of 1 as follows :

Δx	$1 + \Delta x$	$f(1)$	$f(1 + \Delta x)$	Δy	$\Delta y / \Delta x$
0.1	1.1	1	1.21	0.21	2.1
0.01	1.01	1	1.0201	1.0201	2.01
0.001	1.001	1	1.002001	0.002001	2.001
– 0.1	0.9	1	0.81	– 0.19	1.9
– 0.01	0.99	1	0.9801	– 0.0199	1.99
– 0.001	0.999	1	0.998001	– 0.001999	1.999

Clearly, $\lim\limits_{\Delta x \to 0^-} \dfrac{\Delta y}{\Delta x} = 2$; $\lim\limits_{\Delta x \to 0^+} \dfrac{\Delta y}{\Delta x} = 2.$

This shows that $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x} = 2$.

Thus the slope of the tangent line to the graph of $y = x^2$ at (1, 1) is $m_{\tan} = 2$.

On the basis of the Fig.10.6 to 10.13, Illustration 10.1, and our intuition, we are prompted to say that if a graph of a function $y = f(x)$ has a tangent line L at a point P, then L must be the line that is the **limit** of the secants PQ through P and Q as $Q \to P$ $(\Delta x \to 0)$. Moreover, the slope $m_{\tan}$ of L should be the **limiting** value of the values $m_{\sec}$ as $\Delta x \to 0$. This is summarised as follows:

Definition 10.1 (Tangent line with slope *m*)

Let f be defined on an open interval containing x_0, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = m_{\tan}$$

exists, then the line passing through $(x_0, f(x_0))$ with slope m is the **tangent line** to the graph of f at the point $(x_0, f(x_0))$.

The slope of the tangent line at $(x_0, f(x_0))$ is also called the **slope of the curve** at that point.

The definition implies that if a graph admits tangent line at a point $(x_0, f(x_0))$ then it is unique since a point and a slope determine a single line.

The conditions of the definition can be formulated in 4 steps :

(i) Evaluate f at x_0 and $x_0 + \Delta x$: $f(x_0)$ and $f(x_0 + \Delta x)$

(ii) Compute Δy : $\Delta y = f(x_0 + \Delta x) - f(x_0)$

(iii) Divide Δy by Δx ($\neq 0$) : $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$

(iv) Compute the limit as $\Delta x \to 0 (\neq 0)$: $m_{\tan} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

The computation of the slope of the graph in the Illustration 10.1 can be facilitated using the definitions.

(i) $f(1) = 1^2 = 1$. For any $\Delta x \neq 0$

$$f(1 + \Delta x) = (1 + \Delta x)^2 = 1 + 2\Delta x + (\Delta x)^2$$

(ii) $$\Delta y = f(1 + \Delta x) - f(1) = 2\Delta x + (\Delta x)^2 = \Delta x\,(2 + \Delta x)$$

(iii) $$\frac{\Delta y}{\Delta x} = \frac{\Delta x(2+\Delta x)}{\Delta x} = 2 + \Delta x .$$

(iv) $$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(2+\Delta x) = 2+0 = 2 .$$

Example 10.1

Find the slope of the tangent line to the graph of $f(x) = 7x + 5$ at any point $(x_0, f(x_0))$.

Solution

Step (i) $f(x_0) = 7x_0 + 5$.

For any $\Delta x \neq 0$,

$$f(x_0 + \Delta x) = 7(x_0 + \Delta x) + 5 = 7x_0 + 7\Delta x + 5$$

Step (ii) $$\Delta y = f(x_0 + \Delta x) - f(x_0) = (7x_0 + 7\Delta x + 5) - (7x_0 + 5) = 7\Delta x$$

Step (iii) $$\frac{\Delta y}{\Delta x} = 7$$

Thus, at any point on the graph of $f(x) = 7x + 5$, we have

Step (iv) $$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}(7) = 7.$$

Note that for a linear graph, $\dfrac{\Delta y}{\Delta x}$ is a constant, depends neither on x_0 nor on the increment Δx.

Example 10.2

Find the slope of tangent line to the graph of $f(x) = -5x^2 + 7x$ at $(5, f(5))$.

Solution

Step (i) $f(5) = -5(5)^2 + 7 \times 5 = -125 + 35 = -90$.

For any $\Delta x \neq 0$,

$$f(5+\Delta x) = -5(5+\Delta x)^2 + 7(5+\Delta x) = -90 - 43\Delta x - 5(\Delta x)^2.$$

Step (ii)

$$\begin{aligned} \Delta y &= f(5+\Delta x) - f(5) \\ &= -90 - 43\Delta x - 5(\Delta x)^2 + 90 = -43\Delta x - 5(\Delta x)^2 \\ &= \Delta x\,[-43 - 5\Delta x]. \end{aligned}$$

Step (iii)

$$\frac{\Delta y}{\Delta x} = -43 - 5\Delta x$$

Step (iv)

$$m_{\tan} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -43.$$

10.2.2 Velocity of Rectilinear motion

Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = t_0$ to $t = t_0 + \Delta t$, the change in position is $f(t_0 + \Delta t) - f(t_0)$. The average velocity over this time interval is

$$v_{\text{avg}} = \frac{\text{displacement}}{\text{time}} = \frac{f(t_0+\Delta t) - f(t_0)}{\Delta t} = \frac{\text{Change in } s}{\text{Change in } t} = \frac{\Delta s}{\Delta t}$$ which is same as the slope of the secant line PQ in fig. 10.16.

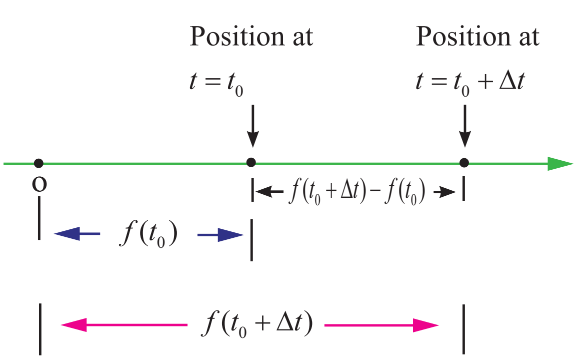


Fig. 10.15

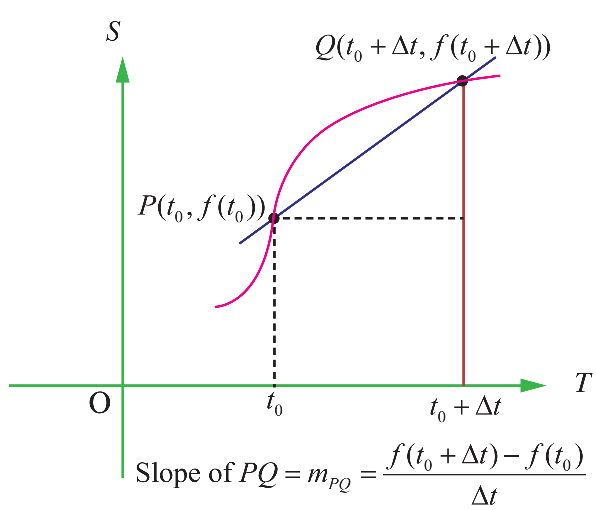


Fig. 10.16

In this time interval Δt (from t_0 to $t_0 + \Delta t$) the motion may be of entirely different types for the same distance covered (traversed). This is illustrated graphically by the fact that we can draw entirely different curves C_1, C_2, C_3 ... between the points P and Q in the plane. These curves are the graphs of quite different motions in the given time intervals, all the motions having the same average velocity $\frac{\Delta s}{\Delta t}$.

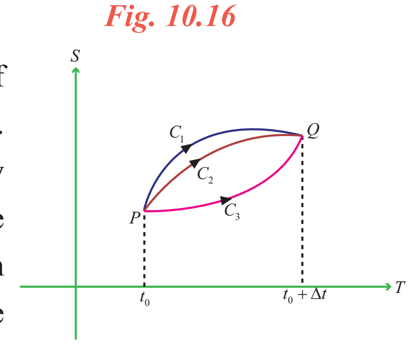


Fig. 10.17

Now suppose we compute the average velocities over shorter and shorter time intervals $[t_0, t_0 + \Delta t]$. In other words, we let Δt approach 0. Then we define the velocity (or instantaneous velocity) $v(t_0)$ at time $t = t_0$ as the limit of these average velocities.

$$v(t_0) = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}.$$

This means that the velocity at time $t = t_0$ is equal to the slope of the tangent line at P.

Illustration 10.2

The distance s travelled by a body falling freely in a vacuum and the time t of descent are variables. They depend on each other. This dependence is expressed by the law of the free fall :

$$s = \frac{1}{2} g t^2 \text{ (absence of initial velocity), } g \text{ is the gravitational constant.}$$

Step (i) $\quad f(t_0 + \Delta t) = \frac{1}{2} g (t_0 + \Delta t)^2 = \frac{1}{2} g \left(t_0^2 + 2 t_0 \Delta t + (\Delta t)^2\right)$

Step (ii) $\quad \Delta s = f(t_0 + \Delta t) - f(t_0)$

$$= \frac{1}{2} g \left[(t_0^2 + 2 t_0 \Delta t + (\Delta t)^2 \right] - \frac{1}{2} g t_0^2$$

$$= g \, \Delta t \left[t_0 + \frac{1}{2} \Delta t \right]$$

Step (iii) $\quad \frac{\Delta s}{\Delta t} = \frac{g \, \Delta t \left[t_0 + \frac{1}{2} \Delta t \right]}{\Delta t} = g \left[t_0 + \frac{1}{2} \Delta t \right]$

Step (iv) $\quad v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = g t_0.$

It is clear from this that the velocity is completely defined by the instant t_0. It is proportional to the time of motion (of the fall).

10.2.3 The derivative of a Function

We have now arrived at a crucial point in the study of calculus. The limit used to define the slope of a tangent line or the instantaneous velocity of a freely falling body is also used to define one of the two fundamental operations of calculus – **differentiation**.

Definition 10.2

Let f be defined on an open interval $I \subseteq \mathbb{R}$ containing the point x_0, and suppose that $\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists. Then f is said to be **differentiable** at x_0 and the **derivative** of f at x_0, denoted by $f'(x_0)$, is given by

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

For all x for which this limit exists,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \text{ is a function of } x.$$

BCJNQ9

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation**. A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval** (a, b) if it is differentiable at every point in (a, b).

In addition to $f'(x)$, which is read as 'f prime of x' or 'f dash of x', other notations are used to denote the derivative of $y = f(x)$. The most common notations are

$$f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], D_x[y] \text{ or } Dy \text{ or } y_1. \text{ Here } \frac{d}{dx} \text{ or } D \text{ is the differential operator.}$$

The notation $\frac{dy}{dx}$ is read as "derivative of y with respect to x" or simply "$dy - dx$", or we should rather read it as "*Dee y Dee x*" or "*Dee Dee x* of *y*". But it is cautioned that we should not regard $\frac{dy}{dx}$ as the quotient $dy \div dx$ and should not refer it as "*dy by dx*". The symbol $\frac{dy}{dx}$ is known as **Leibnitz symbol.**

10.2.4 One sided derivatives (left hand and right hand derivatives)

For a function $y = f(x)$ defined in an open interval (a, b) containing the point x_0, the **left hand and right hand derivatives** of f at $x = x_0$ are respectively denoted by $f'(x_0^-)$ and $f'(x_0^+)$, are defined as

$$f'(x_0^-) = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$\text{and } f'(x_0^+) = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \text{ provided the limits exist.}$$

That is, the function is differentiable from the left and right. As in the case of the existence of limits of a function at x_0, it follows that $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists if and only if both $f'(x_0^-) = \lim_{\Delta x \to 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ and $f'(x_0^+) = \lim_{\Delta x \to 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exist and $f'(x_0^-) = f'(x_0^+)$.

Therefore $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ if and only if $f'(x_0^-) = f'(x_0^+)$.

If any one of the condition fails then f is not differentiable at x_0.

In terms of $h = \Delta x > 0$,

$$f'(x_0^+) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \text{ and}$$

$$f'(x_0^-) = \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{h}.$$

Definition 10.3

A function f is said to be **differentiable in the closed interval** $[a,b]$ if it is differentiable on the open interval (a,b) and at the end points a and b,

$$f'(a) = \lim_{\Delta x \to 0^+} \frac{f(a+\Delta x)-f(a)}{\Delta x} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}, \quad h>0$$

$$f'(b) = \lim_{\Delta x \to 0^-} \frac{f(b+\Delta x)-f(b)}{\Delta x} = \lim_{h\to 0}\frac{f(b-h)-f(b)}{-h}, \quad h>0.$$

If f is differentiable at $x = x_0$, then $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$, where $x = x_0 + \Delta x$ and $\Delta x \to 0$ is equivalent to $x \to x_0$. This alternative form is some times more convenient to be used in computations.

As a matter of convenience, if we let $h = \Delta x$, then $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$, provided the limit exists.

10.3 Differentiability and Continuity

Illustration 10.3

Test the differentiability of the function $f(x) = |x-2|$ at $x = 2$.

Solution

We know that this function is continuous at $x = 2$.

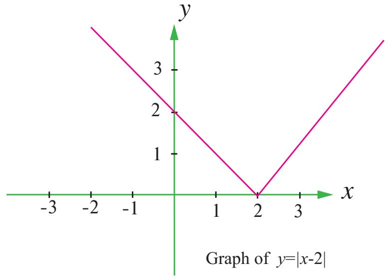


Fig. 10.18

But $$f'(2^-) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-}\frac{|x-2|-0}{x-2}$$

$$= \lim_{x\to 2^-}\frac{|x-2|}{x-2} = \lim_{x\to 2^-}\frac{-(x-2)}{(x-2)} = -1 \text{ and}$$

$$f'(2^+) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+}\frac{|x-2|-0}{x-2} = \lim_{x\to 2^+}\frac{(x-2)}{(x-2)} = 1$$

Since the one sided derivatives $f'(2^-)$ and $f'(2^+)$ are not equal, $f'(2)$ does not exist. That is, f is not differentiable at $x = 2$. At all other points, the function is differentiable.

If $x_0 \neq 2$ is any other point then

$$f'(x_0) = \lim_{x\to x_0}\frac{|x-x_0|}{x-x_0} = \begin{cases} 1 & \text{if } x > x_0 \\ -1 & \text{if } x < x_0 \end{cases}$$

Thus $$f'(2) = \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

The fact that $f'(2)$ does not exist is reflected geometrically in the fact that the curve $y = |x-2|$ does not have a tangent line at (2, 0). Note that the curve has a sharp edge at (2, 0).

Illustration 10.4

Examine the differentiability of $f(x) = x^{\frac{1}{3}}$ at $x = 0$.

Solution

Let $f(x) = x^{\frac{1}{3}}$. Clearly, there is no hole (or break) in the graph of this function and hence it is continuous at all points of its domain.

Let us check whether $f'(0)$ exists.

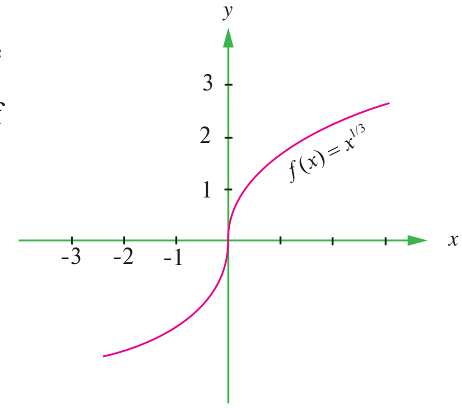


Fig. 10.19

Now $\quad f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{x^{\frac{1}{3}}-0}{x}$

$$= \lim_{x\to 0} x^{\frac{-2}{3}} = \lim_{x\to 0}\frac{1}{x^{\frac{2}{3}}} \to \infty$$

Therefore, the function is not differentiable at $x = 0$. From the Fig. 10.19, further we conclude that the tangent line is vertical at $x = 0$. So f is not differentiable at $x = 0$.

If a function is continuous at a point, then it is not necessary that the function is differentiable at that point.

Example 10.3

Show that the greatest integer function $f(x) = \lfloor x \rfloor$ is not differentiable at any integer?

Solution

The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous at every integer point n, since $\lim\limits_{x\to n^-}\lfloor x \rfloor = n-1$ and $\lim\limits_{x\to n^+}\lfloor x \rfloor = n$. Thus $f'(n)$ does not exist.

What can you say about the differentiability of this function at other points?

Illustration 10.5

Let $f(x) = \begin{cases} x & x \le 0 \\ 1+x & x > 0 \end{cases}$

Compute $f'(0)$ if it exists.

Solution

Look at the graph drawn.

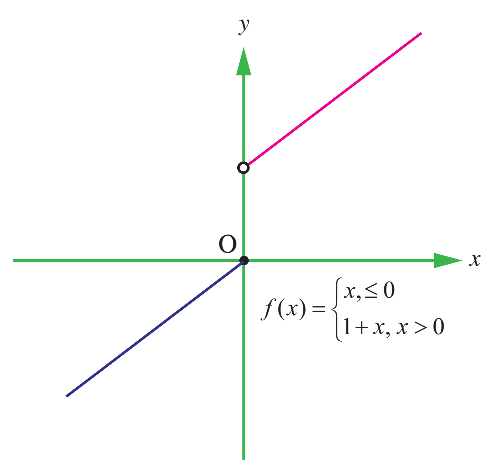


Fig. 10.20

$$f'(0^-) = \lim_{\Delta x\to 0^-}\frac{f(0+\Delta x)-f(0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0^-}\frac{f(\Delta x)}{\Delta x}$$

$$f'(0^-) = \lim_{\Delta x\to 0^-}\frac{\Delta x}{\Delta x} = 1$$

$$f'(0^+) = \lim_{\Delta x\to 0^+}\frac{1+\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0^+} \left(1 + \frac{1}{\Delta x}\right) \to \infty$$

Therefore $f'(0)$ does not exist.

Here we observe that the graph of f has a jump at $x = 0$. That is $x = 0$ is a jump discontinuity.

The above illustrations and examples can be summarised to have the following conclusions.

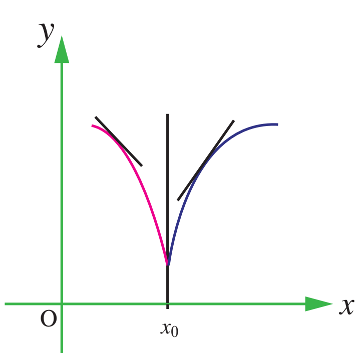
A function f is not differentiable at a point x_0 belonging to the domain of f if one of the following situations holds:

(i) f has a vertical tangent at x_0.

(ii) The graph of f comes to a point at x_0 (either a sharp edge $\vee$ or a sharp peak $\wedge$)

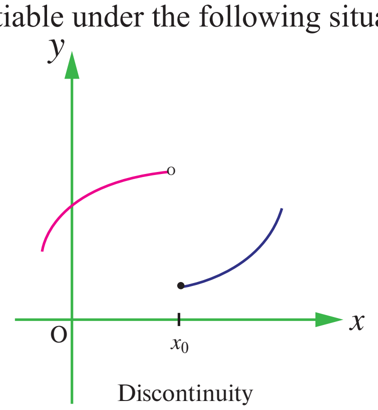
(iii) f is discontinuous at x_0.

A function fails to be differentiable under the following situations :



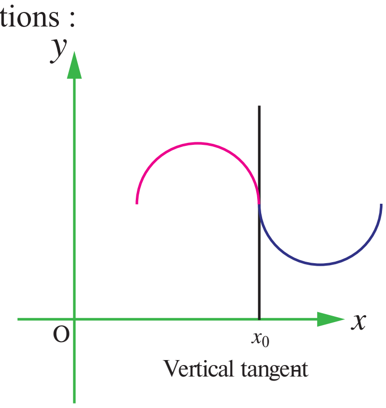
Vertical tangent

Fig. 10.21



Discontinuity

Fig. 10.22



Vertical tangent

Fig. 10.23

We have seen in illustration 10.3 and 10.4, the function $f(x) = |x-2|$ and $f(x) = x^{\frac{1}{3}}$ are respectively continuous at $x = 2$ and $x = 0$ but not differentiable there, whereas in Example 10.3 and Illustration 10.5, the functions $f(x) = \lfloor x \rfloor$ and $f(x) = \begin{cases} x & x \le 0 \\ 1+x & x > 0 \end{cases}$ are respectively not continuous at any integer $x = n$ and $x = 0$ respectively and not differentiable too. The above argument can be condensed and encapsuled to state: **Discontinuity implies non-differentiability.**

Theorem 10.1 (Differentiability implies continuity)

If f is differentiable at a point $x = x_0$, then f is continuous at x_0.

Proof

Let $f(x)$ be a differentiable function on an interval (a, b) containing the point x_0. Then $f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ exists, in the sense that $f'(x_0)$ is a unique real number.

$$\text{Now } \lim_{\Delta x \to 0} \left[f(x_0 + \Delta x) - f(x_0)\right] = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \times \Delta x$$

$$= \lim_{\Delta x \to 0}\left[\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right]\times \lim_{\Delta x\to 0}(\Delta x)$$

$$= f'(x_0)\times 0 = 0 .$$

This implies, f is continuous at $x = x_0$.

Derivatives from first principle

The process of finding the derivative of a function using the conditions stated in the definition of derivatives is known as derivatives from first principle.

EXERCISE 10.1

(1) Find the derivatives of the following functions using first principle.

(i) $f(x)=6$ (ii) $f(x) = -4x + 7$ (iii) $f(x) = -x^2 + 2$

(2) Find the derivatives from the left and from the right at $x = 1$ (if they exist) of the following functions. Are the functions differentiable at $x = 1$?

(i) $f(x) = |x - 1|$ (ii) $f(x)=\sqrt{1-x^2}$ (iii) $f(x) = \begin{cases} x, & x\le 1 \\ x^2, & x>1 \end{cases}$

(3) Determine whether the following function is differentiable at the indicated values.

(i) $f(x) = x\,|\,x\,|$ at $x = 0$ (ii) $f(x) = |\,x^2-1\,|$ at $x=1$

(iii) $f(x) = |x| + |x - 1|$ at $x = 0, 1$ (iv) $f(x)=\sin|\,x\,|$ at $x=0$

(4) Show that the following functions are not differentiable at the indicated value of x.

(i) $f(x) = \begin{cases} -x+2, & x\le 2 \\ 2x-4, & x>2 \end{cases}$; $x=2$ (ii) $f(x) = \begin{cases} 3x, & x<0 \\ -4x, & x\ge 0 \end{cases}$, $x=0$

(5) The graph of f is shown below. State with reasons that x values (the numbers), at which f is not differentiable.

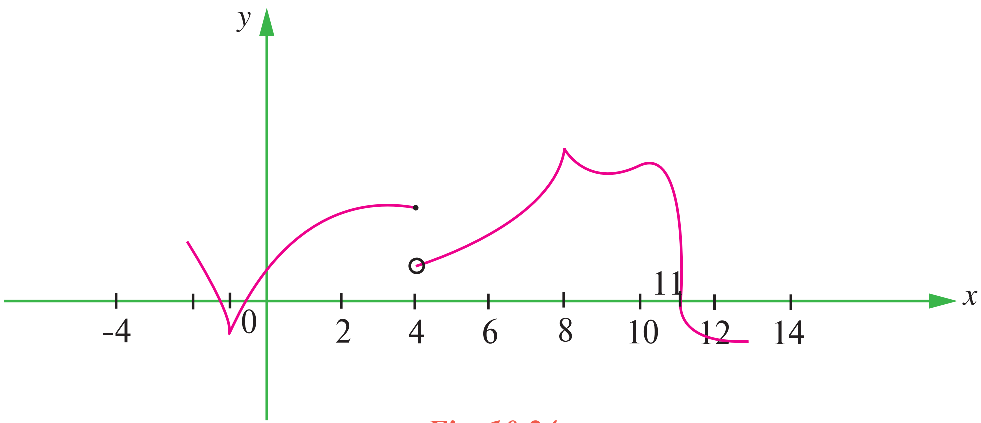


Fig. 10.24

(6) If $f(x) = |x + 100| + x^2$, test whether $f'(-100)$ exists.

(7) Examine the differentiability of functions in $\mathbb{R}$ by drawing the diagrams.

(i) $|\sin x|$ (ii) $|\cos x|$.

10.4 Differentiation Rules

If f is a real valued function of a real variable defined on an open interval I and if $y = f(x)$ is a differentiable function of x, then $\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$. In general finding such direct derivatives using first principle is extremely laborious and difficult operation in the majority of cases. But if we know, once and for all, the derivatives of all the basic elementary functions, together with rules of differentiating the algebra of functions and functions of a function, we can find the derivative of any element – any function without carrying out limit process each time. Hence the operation of differentiation can be made automatic for the class of functions that concern us.

Now we divert our attention to the rules for differentiation of a sum, product and quotient.

Theorem 10.2

The derivative of the sum of two (or more) differentiable functions is equal to the sum of their derivatives. That is, if u and v are two differentiable functions then $\boxed{\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v}$

Proof

Let u and v be two real valued functions defined and differentiable on an open interval $I \subseteq \mathbb{R}$. Let $y = u + v$, then $y = f(x)$ is a function defined on I, and by hypothesis

$$u'(x) = \frac{du}{dx} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x}$$

$$v'(x) = \frac{dv}{dx} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x) - v(x)}{\Delta x} \text{ exist.}$$

Now, $$f(x+\Delta x) = u(x+\Delta x) + v(x+\Delta x)$$

$$f(x+\Delta x) - f(x) = u(x+\Delta x) - u(x) + v(x+\Delta x) - v(x).$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{u(x+\Delta x) - u(x)}{\Delta x} + \frac{v(x+\Delta x) - v(x)}{\Delta x}.$$

This implies, $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{(x+\Delta x) - v(x)}{\Delta x}.$$

That is, $$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = u'(x) + v'(x).$$

That is, $$f'(x) = u'(x) + v'(x).$$

or $$\left(u+v\right)'(x) = u'(x) + v'(x).$$

That is, $$\boxed{\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v}.$$

This can be extended to finite number of differentiable functions $u_1, u_2, \ldots, u_n$ and $(u_1 + u_2 + \ldots + u_n)' = u_1' + u_2' + \ldots + u_n'$.

Theorem 10.3

Let u and v be two differentiable functions. Then $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.

Proof

Let u and v be the given two differentiable functions so that

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x)-u(x)}{\Delta x} = \frac{du}{dx} \text{ and } \lim_{\Delta x \to 0} \frac{v(x+\Delta x)-v(x)}{\Delta x} = \frac{dv}{dx}.$$

Let $y = f(x) = u.v$

Then $f(x + \Delta x) = u(x + \Delta x)\, v(x + \Delta x)$, and

$$f(x + \Delta x) - f(x) = u(x + \Delta x)\, v(x + \Delta x) - u(x).v(x)$$

$$= v(x + \Delta x)\,[u(x+\Delta x)-u(x)]+u(x)[v(x+\Delta x)-v(x)].$$

This implies, $$\frac{f(x+\Delta x)-f(x)}{\Delta x} = u(x)\frac{[v(x+\Delta x)-v(x)]}{\Delta x}+v(x+\Delta x)\frac{[u(x+\Delta x)-u(x)]}{\Delta x}.$$

$$\lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = u(x)\lim_{\Delta x \to 0}\frac{v(x+\Delta x)-v(x)}{\Delta x}+\lim_{\Delta x \to 0}v(x+\Delta x)\lim_{\Delta x \to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}$$

$$= u(x)v'(x)+v(x)u'(x) \text{ (since } v \text{ is continuous, } \lim_{\Delta x \to 0} v(x+\Delta x) = v(x)).$$

That is, $$f'(x) = \boxed{\frac{d}{dx}(uv) = u\frac{dv}{dx}+v\frac{du}{dx}}.$$

or $\boxed{(uv)' = uv' + vu'}$

Similarly $(uvw)' = uvw' + uv'w + u'vw$.

This can be extended to a finite number of differentiable functions $u_1, u_2, \ldots\ u_n$, using induction :

$$(u_1.u_2\ldots,u_n)' = u_1u_2\ldots u_{n-1}u_n' + u_1u_2\ldots u_{n-1}'u_n + \cdots + u_1'u_2\cdots u_n.$$

Theorem 10.4 (Quotient Rule)

Let u and v be two differentiable functions with $v(x) \neq 0$. Then $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Proof

Let $y = f(x) = \frac{u}{v}$, u and v are differentiable functions of x and where $v(x) \neq 0$.

Now $$f(x+\Delta x) = \frac{u(x+\Delta x)}{v(x+\Delta x)}$$

This implies, $$f(x+\Delta x) - f(x) = \frac{u(x+\Delta x)}{v(x+\Delta x)} - \frac{u(x)}{v(x)}$$

$$= \frac{v(x)u(x+\Delta x) - u(x)v(x+\Delta x)}{v(x+\Delta x)v(x)}.$$

This implies, $$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\dfrac{v(x)\left[u(x+\Delta x)-u(x)\right]}{\Delta x}-\dfrac{u(x)\left[v(x+\Delta x-v(x)\right]}{\Delta x}}{v(x+\Delta x)v(x)}$$

This implies, $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{v(x)\lim\limits_{\Delta x\to 0}\dfrac{u(x+\Delta x)-u(x)}{\Delta x}-u(x)\lim\limits_{\Delta x\to 0}\dfrac{v(x+\Delta x)-v(x)}{\Delta x}}{v(x)\lim\limits_{\Delta x\to 0}v(x+\Delta x)}$$

$$= \frac{v(x)u'(x)-u(x)v'(x)}{v(x)v(x)} \qquad \left(\because \lim_{\Delta x\to 0}v(x+\Delta x)=v(x)\right)$$

This implies, $$f'(x) = \frac{v(x)u'(x)-u(x)v'(x)}{\left[v(x)\right]^2}.$$

That is, $$\frac{d}{dx}(f) = \frac{vu'-uv'}{v^2}$$

or $$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}}.$$

Theorem 10.5 (Chain Rule / Composite Function Rule or Function of a Function Rule)

Let $y=f(u)$ be a function of u and in turn let $u=g(x)$ be a function of x so that $y=f\left(g(x)\right)=(fog)(x)$. Then $\frac{d}{dx}(f(g(x))=f'(g(x))g'(x)$.

Proof

In the above function $u=g(x)$ is known as the inner function and f is known as the outer function. Note that, ultimately, y is a function of x.

Now $\Delta u = g(x+\Delta x)-g(x)$

Therefore, $$\frac{\Delta y}{\Delta x}=\frac{\Delta y}{\Delta u}\times\frac{\Delta u}{\Delta x}=\frac{f(u+\Delta u)-f(u)}{\Delta u}\times\frac{g(x+\Delta x)-g(x)}{\Delta x}.$$

Note that $\Delta u\to 0$ as $\Delta x\to 0$

Therefore, $$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta u}\times\frac{\Delta y}{\Delta x}\right)$$

$$= \lim_{\Delta u\to 0}\left(\frac{\Delta y}{\Delta u}\right).\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)$$

$$= \lim_{\Delta u\to 0}\frac{f(u+\Delta u)-f(u)}{\Delta u}\times\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$= f'(u)\times u'(x)$$

$$= f'\left(g(x)\right)g'(x) \quad \text{or} \quad \boxed{\frac{d}{dx}\left(f\left(g(x)\right)=f'\left(g(x)\right)g'(x)\right)}.$$

Thus, to differentiate a function of a function $y = f(g(x))$, we have to take the derivative of the outer function f regarding the argument $g(x) = u$, and multiply the derivative of the inner function $g(x)$ with respect to the independent variable x. The variable u is known as **intermediate argument**.

Theorem 10.6

Let $f(x)$ be a differentiable function and let $y = k\,f(x), k \neq 0$. Then $\dfrac{d}{dx}(k\,f(x)) = k\dfrac{d}{dx}f(x)$.

Proof

Since f is differentiable, $\displaystyle\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$.

$$\begin{aligned}
\text{Let } y &= h(x) = k\,f(x)\\
h(x+\Delta x) &= k\,f(x+\Delta x)\\
h(x+\Delta x) - h(x) &= kf(x+\Delta x) - k\,f(x)\\
&= k\left[f(x+\Delta x)-f(x)\right].\\
\frac{h(x+\Delta x)-h(x)}{\Delta x} &= k\frac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}.\\
\text{This implies, } \lim_{\Delta x\to 0}\frac{h(x+\Delta x)-h(x)}{\Delta x} &= \lim_{\Delta x\to 0} k\,\frac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}\\
&= k\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)-f(x)\right]}{\Delta x}\\
&= kf'(x) = k\frac{d}{dx}f(x)\\
\frac{d}{dx}\left(h(x)\right) &= k\frac{d}{dx}f(x).
\end{aligned}$$

That is, $\boxed{\dfrac{d}{dx}\left(kf(x)\right) = k\dfrac{d}{dx}f(x)}$.

10.4.1 Derivatives of basic elementary functions

We shall now find the derivatives of all the basic elementary functions; we start with the constant function.

(1) The derivative of a constant function is zero.

$$\begin{aligned}
\text{Let } y &= f(x) = k, \ k \text{ is a constant.}\\
\text{Then } f(x+\Delta x) &= k \text{ and}\\
f(x+\Delta x) - f(x) &= k - k = 0.\\
\text{This implies, } \frac{f(x+\Delta x)-f(x)}{\Delta x} &= 0\\
\text{This implies, } \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= 0.
\end{aligned}$$

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That is, $f'(x) = 0$.

or $\boxed{\frac{d}{dx}(k)=0}$.

(2) The power function $y = x^n$, $n > 0$ is an integer.

Let $f(x) = x^n$.

Then, $f(x+\Delta x) = (x+\Delta x)^n$ and

$f(x+\Delta x) - f(x) = (x+\Delta x)^n - x^n$.

This implies, $\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \dfrac{(x+\Delta x)^n - x^n}{(x+\Delta x)-x}$

This implies, $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^n - x^n}{(x+\Delta x)-x}$

$= \lim\limits_{y\to x}\dfrac{y^n - x^n}{y-x} = nx^{n-1}$, where $y = x+\Delta x$ and $y \to x$ as $\Delta x \to 0$.

That is, $\dfrac{d}{dx}f(x) = nx^{n-1}$ or $\boxed{\frac{d}{dx}(x^n)=nx^{n-1}}$.

Corollary 10.1

When $n = \dfrac{p}{q}, (p,q)=1$, $\dfrac{d}{dx}\left(x^{\frac{p}{q}}\right) = \dfrac{p}{q}x^{\frac{p}{q}-1}$.

Corollary 10.2

For any real number α, $\dfrac{d}{dx}(x^{\alpha}) = \alpha x^{\alpha-1}$.

For instance,

(1) $\dfrac{d}{dx}(5) = 0$ since 5 is a constant.

(2) $\dfrac{d}{dx}(x^3) = 3x^2$, by power function rule.

(3) $\dfrac{d}{dx}\left(x^{\frac{3}{2}}\right) = \dfrac{3}{2}x^{\frac{3}{2}-1} = \dfrac{3}{2}x^{\frac{1}{2}}$, by power function rule.

(4) $\dfrac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$, by power function rule.

(5) $\dfrac{d}{dx}\left(x^{\frac{2}{3}}\right) = \dfrac{2}{3}x^{\frac{2}{3}-1} = \dfrac{2}{3}x^{\frac{-1}{3}}, (x \neq 0)$, by power function rule.

(6) $\dfrac{d}{dx}(100x^9) = 100\dfrac{d}{dx}(x^9) = 100\times 9x^{9-1} = 900x^8$ by theorem 10.6.

(3) Derivative of the logarithmic function

The natural logarithm of x is denoted by $\log_e x$ or log$x$ or $ln\ x$

$$\text{Let } y = f(x) = \log x$$

$$\text{Now } f(x+\Delta x) = \log(x+\Delta x) \text{ and}$$

$$f(x+\Delta x) - f(x) = \log(x+\Delta x) - \log x$$

$$= \log\left(\frac{x+\Delta x}{x}\right).$$

$$= \log\left(1+\frac{\Delta x}{x}\right).$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x}.$$

$$\text{We know that } \lim_{\alpha\to 0}\frac{\log(1+\alpha k)}{\alpha} = k\lim_{\alpha\to 0}\frac{\log(1+k\alpha)}{k\alpha} = k$$

$$\text{Therefore, } \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\log\left(1+\frac{\Delta x}{x}\right)}{\Delta x} = \frac{1}{x}.$$

$$\text{That is, } \boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}.$$

Corollary 10.3

$$\text{If } y = f(x) = \log_a x \text{ then } f'(x) = \frac{1}{(\log a)x}.$$

$$\text{We have } f(x) = \log_a x = \log_a e \times \log_e x = (\log_a e)\log x$$

$$\text{Therefore, } \frac{d}{dx}\left(f(x)\right) = \frac{d}{dx}(\log_a e \times \log x)$$

$$= (\log_a e)\frac{d}{dx}(\log x) \qquad \text{(by constant multiple rule)}$$

$$= \log_a e.\frac{1}{x} \text{ (or) } \frac{1}{(\log a)x}.$$

(4) Derivative of the exponential function

$$\text{Let } y = a^x.$$

$$\text{Then } f(x+\Delta x) - f(x) = a^{x+\Delta x} - a^x$$

$$= a^x(a^{\Delta x}-1) \text{ and}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = a^x\left(\frac{a^{\Delta x}-1}{\Delta x}\right).$$

We know that $\lim\limits_{\Delta x\to 0}\dfrac{a^{\Delta x}-1}{\Delta x} = \log a$.

Therefore, $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = a^x \lim\limits_{\Delta x\to 0}\left(\dfrac{a^{\Delta x}-1}{\Delta x}\right)$

$$= a^x \times \log a$$

or $\boxed{\dfrac{d}{dx}(a^x) = a^x \log a}$.

In particular, $\dfrac{d}{dx}(e^x) = e^x \log e$

$$\boxed{\frac{d}{dx}(e^x) = e^x}.$$

(5) The derivatives of the Trigonometric functions

(i) The sine function, sin$x$.

Let $y = f(x) = \sin x$.

Then $f(x+\Delta x) = \sin(x+\Delta x)$ and

$$f(x+\Delta x)-f(x) = \sin(x+\Delta x)-\sin x = 2\sin\frac{\Delta x}{2}\cos\left(x+\frac{x}{2}\right).$$

Now $\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \dfrac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}\cos\left(x+\dfrac{\Delta x}{2}\right)$.

This implies, $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sin\left(\frac{\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)}.\lim\limits_{\Delta x\to 0}\cos\left(x+\dfrac{\Delta x}{2}\right)$

$=$ $1 \times \cos x$ (Since $\cos x$ is continuous, $\lim\limits_{\Delta x\to 0}\cos(x+\Delta x) = \cos x$) .

$=$ $\cos x$

That is $\boxed{\dfrac{d}{dx}(\sin x) = \cos x}$.

(ii) The cosine function, $\cos x$

$$\text{Let } y = \cos x = \sin\left(x+\frac{\pi}{2}\right).$$

$$\text{Then, } \frac{dy}{dx} = \frac{d}{dx}\sin\left(x+\frac{\pi}{2}\right)$$

$$\text{Let } u = x+\frac{\pi}{2}$$

$$\frac{du}{dx} = 1 + 0 = 1$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{d}{dx}(\sin u) = \frac{d}{dx}(\sin u)\frac{du}{dx} \text{ by Chain rule}$$

$$= \cos u \times 1 = \cos u = \cos\left(x+\frac{\pi}{2}\right) = -\sin x .$$

That is, $\boxed{\frac{d}{dx}(\cos x) = -\sin x}$.

(iii) The tangent function, $\tan x$

$$\text{Let } y = f(x) = \tan x .$$

$$= \frac{\sin x}{\cos x}$$

$$\text{Therefore, } \frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \quad \text{(by quotient rule)}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} .$$

That is, $\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$.

(iv) The secant function, $\sec x$

$$\text{Let } y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1} .$$

$$\text{Then } \frac{dy}{dx} = (-1)(\cos x)^{-2}(-\sin x) \quad \text{(by chain rule)}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}.\frac{\sin x}{\cos x} = \sec x.\tan x\ .$$

That is, $\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$.

(v) The cosecant function, $\operatorname{cosec} x$

$$\text{Let } y = \operatorname{cosec} x = \frac{1}{\sin x} = (\sin x)^{-1}\ .$$

$$\frac{dy}{dx} = (-1)(\sin x)^{-2}(\cos x) \quad \text{(by chain rule)}$$

$$= -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}.\frac{\cos x}{\sin x} = -\operatorname{cosec} x \cot x\ .$$

That is, $\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$.

(vi) The cotangent function, $\cot x$

$$\text{Let } y = \cot x = \frac{\cos x}{\sin x}.$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x}$$

$$= \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

That is, $\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$.

(6) The derivatives of the inverse trigonometric functions

(i) The derivative of arcsin x or $\sin^{-1} x$

$$\text{Let } y = f(x) = \sin^{-1} x\ .$$

$$\text{Then } y + \Delta y = f(x+\Delta x) = \sin^{-1}(x + \Delta x)$$

This implies, $x = \sin y$ and

$$x + \Delta x = \sin(y + \Delta y).$$

$$\text{Therefore, } \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\sin(y+\Delta y)-\sin y} = \frac{1}{\dfrac{\sin(y+\Delta y)-\sin y}{\Delta y}}$$

As $\Delta x \to 0$, $\Delta y \to 0$ also, so that

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta y\to 0}\frac{1}{\dfrac{\sin(y+\Delta y)-\sin y}{\Delta y}}$$

$$= \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

That is, $\boxed{\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}}$.

(ii) The derivative of arccos x or $\cos^{-1} x$

We know the identity :

$$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

$$\text{This implies, } \frac{d}{dx}(\sin^{-1}x+\cos^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$$

$$\text{This implies, } \frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\cos^{-1}x) = 0$$

$$\text{Therefore, } \frac{1}{\sqrt{1-x^2}} + \frac{d}{dx}(\cos^{-1}x) = 0$$

or $\boxed{\dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1-x^2}}}$.

(iii) The derivative of arctan x or $\tan^{-1} x$

$$\text{Let } y = f(x) = \tan^{-1}x. \quad \text{... (1)}$$

$$\text{This implies, } y + \Delta y = f(x+\Delta x) = \tan^{-1}(x+\Delta x) \quad \text{... (2)}$$

$$x = \tan y \text{ and}$$

$$x + \Delta x = \tan(y + \Delta y)$$

This implies, $\Delta x = \tan(y + \Delta y) - \tan y$

$$\text{Therefore, } \frac{\Delta y}{\Delta x} = \frac{\Delta y}{\tan(y+\Delta y)-\tan y}$$

$$= \frac{1}{\dfrac{\tan(y+\Delta y)-\tan y}{\Delta y}}.$$

As $\Delta x \to 0$, $\Delta y \to 0$ also, so that

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{1}{\dfrac{\lim\limits_{\Delta y\to 0}\tan(y+\Delta y)-\tan y}{\Delta y}} = \frac{1}{\dfrac{d}{dy}(\tan y)}$$

$$= \frac{1}{\sec^2 y}$$

$$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

That is, $\boxed{\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}}$.

(iv) The derivative of arccot x or $\cot^{-1} x$

We know the identity

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}.$$

$$\text{This implies, } \frac{d}{dx}(\tan^{-1} x + \cot^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2}\right) = 0$$

$$\text{This implies, } \frac{d}{dx}(\tan^{-1} x) + \frac{d}{dx}(\cot^{-1} x) = 0$$

$$\text{That is, } \frac{d}{dx}(\cot^{-1} x) = -\frac{d}{dx}(\tan^{-1} x) = -\frac{1}{1+x^2}$$

That is, $\boxed{\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}}$.

(v) The derivative of arcsec x or $\boxed{\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}}$.

(vi) The derivative of arccosec x or $\boxed{\dfrac{d}{dx}(\text{cosec}^{-1} x) = \dfrac{-1}{x\sqrt{x^2-1}}}$.

The proofs of (v) and (vi) are left as exercises.

Example 10.7

Differentiate the following with respect to x :

(i) $y = x^3 + 5x^2 + 3x + 7$ (ii) $y = e^x + \sin x + 2$

(iii) $y = 4\operatorname{cosec} x - \log x - 2e^x$ (iv) $y = \left(x - \frac{1}{x}\right)^2$ (v) $y = xe^x \log x$

(vi) $y = \frac{\cos x}{x^3}$ (vii) $y = \frac{\log x}{e^x}$

(viii) Find $f'(3)$ and $f'(5)$ if $f(x) = |x-4|$.

Solution

(i) $\frac{dy}{dx} = 3x^2 + 10x + 3.$ (ii) $\frac{dy}{dx} = e^x + \cos x.$

(iii) $\frac{dy}{dx} = -4 \operatorname{cosec} x.\cot x - \frac{1}{x} - 2e^x.$

(iv) $y = x^2 + \frac{1}{x^2} - 2 = x^2 + x^{-2} - 2$

$$\frac{dy}{dx} = 2x - 2x^{-2-1} = 2x - \frac{2}{x^3}.$$

(v) $\frac{dy}{dx} = xe^x\left(\frac{1}{x}\right) + e^x.\log x(1) + x\log x(e^x)$

$$= e^x + e^x \log x + xe^x \log x = e^x(1 + \log x + x\log x).$$

(vi) $y = \frac{\cos x}{x^3}$

$$\frac{dy}{dx} = \frac{x^3(-\sin x) - \cos x(3x^2)}{x^6} = \frac{-x^2(x\sin x + 3\cos x)}{x^6} = -\frac{(x\sin x + 3\cos x)}{x^4}.$$

(vii) $y = \frac{\log x}{e^x} = e^{-x}.\log x$

$$\frac{dy}{dx} = e^{-x}\left(\frac{1}{x}\right) + \log x(e^{-x})(-1)$$

$$= e^{-x}\left[\frac{1}{x} - \log x\right].$$

(viii) $f(x) = |x-4| = \begin{cases} -(x-4) & ; \ x < 4 \\ (x-4) & ; \ x \geq 4 \end{cases}$

$$f'(x) = \begin{cases} -1 \text{ if } x < 4 \\ +1 \text{ if } x \geq 4 \end{cases}$$

Therefore, $f'(3) = -1$

$f'(5) = 1$.

EXERCISE 10.2

Find the derivatives of the following functions with respect to corresponding independent variables:

(1) $f(x) = x - 3\sin x$

(2) $y = \sin x + \cos x$

(3) $f(x) = x\sin x$

(4) $y = \cos x - 2\tan x$

(5) $g(t) = t^3\cos t$

(6) $g(t) = 4\sec t + \tan t$

(7) $y = e^x \sin x$

(8) $y = \dfrac{\tan x}{x}$

(9) $y = \dfrac{\sin x}{1+\cos x}$

(10) $y = \dfrac{x}{\sin x + \cos x}$

(11) $y = \dfrac{\tan x - 1}{\sec x}$

(12) $y = \dfrac{\sin x}{x^2}$

(13) $y = \tan\theta(\sin\theta + \cos\theta)$

(14) $y = \operatorname{cosec} x \,.\, \cot x$

(15) $y = x\sin x\cos x$

(16) $y = e^{-x}.\log x$

(17) $y = (x^2+5)\log(1+x)e^{-3x}$

(18) $y = \sin x^\circ$

(19) $y = \log_{10} x$

(20) Draw the function $f'(x)$ if $f(x) = 2x^2 - 5x + 3$

10.4.2 Examples on Chain Rule

Example 10.8

Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Solution

$$\text{Take } u = g(x) = x^2+1 \text{ and } f(u) = \sqrt{u}$$

$$\therefore F(x) = (fog)(x) = f(g(x))$$

$$\text{Since } f'(u) = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} \text{ and}$$

$$g'(x) = 2x, \text{ we get}$$

$$F'(x) = f'(g(x))g'(x)$$

$$= \frac{1}{2\sqrt{x^2+1}}.2x = \frac{x}{\sqrt{x^2+1}}.$$

Example 10.9

Differentiate : (i) $y = \sin(x^2)$ (ii) $y = \sin^2 x$

Solution

(i) The outer function is the sine function and the inner function is the squaring function.

$$\text{Let } u = x^2$$

$$\text{That is, } y = \sin u.$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx}$$

$$= \cos u \times (2x)$$
$$= \cos(x^2).2x$$
$$= 2x\cos(x^2).$$

(ii) $u = \sin x$

Then, $y = u^2$

and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$$= 2u \times \cos x$$
$$= 2\sin x \,.\, \cos x$$
$$= \sin 2x.$$

Example 10.10

Differentiate : $y = (x^3 - 1)^{100}$.

Solution

Take $u = x^3 - 1$ so that

$y = u^{100}$

and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$$= 100u^{100-1} \times (3x^2 - 0)$$
$$= 100(x^3 - 1)^{99} \times 3x^2$$
$$= 300x^2(x^3 - 1)^{99} .$$

Example 10.11

Find $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$.

Solution

First we write : $f(x) = (x^2 + x + 1)^{\frac{-1}{3}}$

Then, $f'(x) = -\frac{1}{3}(x^2 + x + 1)^{\frac{-1}{3} - 1} \frac{d}{dx}(x^2 + x + 1)$
$$= -\frac{1}{3}(x^2 + x + 1)^{\frac{-4}{3}} \times (2x + 1)$$
$$= -\frac{1}{3}(2x + 1)(x^2 + x + 1)^{\frac{-4}{3}} .$$

Example 10.12

Find the derivative of the function $g(t) = \left(\frac{t-2}{2t+1}\right)^9$.

Solution

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^{9-1} \frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \left[\frac{(2t+1)\frac{d}{dt}(t-2)-(t-2)\frac{d}{dt}(2t+1)}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \left[\frac{(2t+1)\times 1-(t-2)\times 2}{(2t+1)^2}\right]$$

$$= 9\left(\frac{t-2}{2t+1}\right)^8 \left[\frac{2t+1-2t+4}{(2t+1)^2}\right]$$

$$= \frac{45(t-2)^8}{(2t+1)^{10}}\ .$$

Example 10.13

Differentiate $(2x+1)^5\ (x^3-x+1)^4$.

Solution

Let $y = (2x+1)^5\ (x^3-x+1)^4$

Take $u = 2x+1$; $v = x^3 - x + 1$ so that

$y = u^5 \,.\, v^4$

$$\frac{dy}{dx} = u^5 . \frac{d}{dx}(v^4) + v^4 \frac{d}{dx}(u^5) \qquad \text{by Product Rule}$$

$$= u^5 .4v^{4-1}\frac{dv}{dx} + v^4 .5u^{5-1}\frac{du}{dx} \qquad \text{by Chain Rule}$$

$$= 4u^5 .v^3 \times (3x^2-1) + 5v^4u^4 \times 2$$

$$= 4(2x+1)^5(x^3-x+1)^3(3x^2-1) + 10(x^3-x+1)^4(2x+1)^4$$

$$= (2x+1)^4(x^3-x+1)^3\left[4(2x+1)(3x^2-1)+10(x^3-x+1)\right]$$

$$= 2(2x+1)^4(x^3-x+1)^3(17x^3+6x^2-9x+3)\,.$$

Example 10.14

Differentiate : $y = e^{\sin x}$.

Solution

Take $u = \sin x$ so that

$y = e^u$

$$\frac{dy}{dx} = \frac{d(e^u)}{du}\times\frac{du}{dx} = e^u \times \cos x = \cos x\, e^{\sin x}.$$

Example 10.15

Differentiate 2^x.

Solution

Let $y = 2^x = e^{x\log 2}$.

Take $u = (\log 2)x$ so that

$$y = e^u$$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = e^u \times \log 2 = \log 2\, e^{x \log 2}$$

$$= (\log 2)2^x.$$

By using the differentiation formula for a^x, one can find the derivative directly.

Example 10.16

If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find y'.

Solution

$$y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$$

Let $\frac{1+x}{1-x} = t$

Then, $y = \tan^{-1} t$

$$\frac{dy}{dx} = \frac{d}{dt}(\tan^{-1} t).\frac{dt}{dx}$$

$$= \frac{1}{1+t^2}.\frac{(1-x).1-(1+x)(-1)}{(1-x)^2}$$

$$= \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}.\frac{(1-x)+(1+x)}{(1-x)^2} = \frac{1}{1+x^2}.$$

EXERCISE 10.3

Differentiate the following :

(1) $y = (x^2+4x+6)^5$
(2) $y = \tan 3x$
(3) $y = \cos(\tan x)$
(4) $y = \sqrt[3]{1+x^3}$
(5) $y = e^{\sqrt{x}}$
(6) $y = \sin(e^x)$
(7) $F(x) = (x^3+4x)^7$
(8) $h(t) = \left(t-\frac{1}{t}\right)^{\frac{3}{2}}$
(9) $f(t) = \sqrt[3]{1+\tan t}$
(10) $y = \cos(a^3+x^3)$
(11) $y = e^{-mx}$
(12) $y = 4\sec 5x$
(13) $y = (2x-5)^4(8x^2-5)^{-3}$
(14) $y = (x^2+1)\sqrt[3]{x^2+2}$
(15) $y = xe^{-x^2}$
(16) $s(t) = \sqrt[4]{\frac{t^3+1}{t^3-1}}$
(17) $f(x) = \frac{x}{\sqrt{7-3x}}$
(18) $y = \tan(\cos x)$
(19) $y = \frac{\sin^2 x}{\cos x}$
(20) $y = 5^{\frac{-1}{x}}$
(21) $y = \sqrt{1+2\tan x}$

(22) $y = \sin^3 x + \cos^3 x$ (23) $y = \sin^2(\cos kx)$ (24) $y = (1+\cos^2 x)^6$

(25) $y = \dfrac{e^{3x}}{1+e^x}$ (26) $y = \sqrt{x+\sqrt{x}}$ (27) $y = e^{x\cos x}$

(28) $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$ (29) $y = \sin\left(\tan\left(\sqrt{\sin x}\right)\right)$ (30) $y = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

10.4.3 Implicit Differentiation

A function in which the dependent variable is expressed solely in terms of the independent variable x, namely, $y = f(x)$, is said to be an **explicit** function. For instance, $y = \frac{1}{2}x^3 - 1$ is an explicit function, whereas an equivalent equation $2y - x^3 + 2 = 0$ is said to define the function **implicitly** or y is an **implicit function** of x.

Now, as we know, the equation

$$x^2 + y^2 = 4 \qquad (1)$$

describes *a* a circle of radius 2 centered at the origin. Equation (1) is not a function since for any choice of x in the interval $-2 < x < 2$ there correspond two values of y, namely

$$f(x) = \sqrt{4-x^2}, -2 \le x \le 2 \qquad (2)$$

$$g(x) = -\sqrt{4-x^2}, -2 \le x \le 2 \qquad (3)$$

(2) represents the top half of the circle (1) and (3) represents the bottom half of the circle (1). By considering either the top half or bottom half, of the circle, we obtain a function. We say that (1) defines at least two implicit functions of x on the interval $-2 \le x \le 2$.

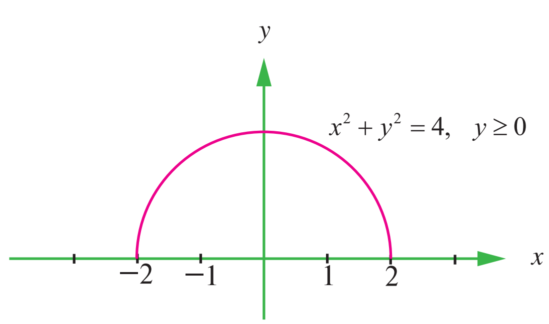

Fig. 10.25

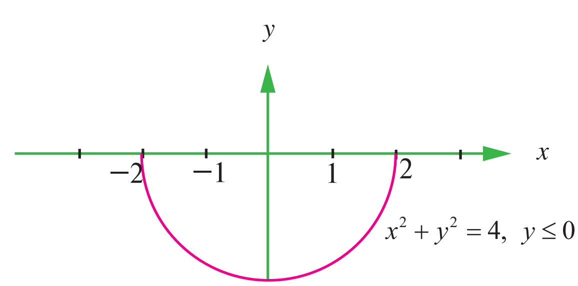

Fig. 10.26

Note that both equations

$x^2 + [f(x)]^2 = 4$ and $x^2 + [g(x)]^2 = 4$ are identities on the interval $-2 \le x \le 2$.

In general, if an equation $F(x, y) = 0$ defines a function f implicitly on some interval, then $F(x, f(x)) = 0$ is an identity on the interval. The graph of f is a portion (or all) of the graph of the equation $F(x, y) = 0$.

A more complicated equation such as $x^4 + x^2y^3 - y^5 = 2x + 1$ may determine several implicit functions on a suitably restricted interval of the x-axis and yet it may not be possible to solve for y in terms of x. However, in some cases we can determine the derivative $\frac{dy}{dx}$ by a process known as **implicit differentiation**. This process consists of differentiating both sides of an equation with respect to x, using the rules of differentiation and then solving for $\frac{dy}{dx}$. Since we think of y as being determined by the given equation as a differentiable function, the chain rule, in the form of the power rule for functions, gives the result.

$$\frac{d}{dx}(y^n) = n\,y^{n-1}\frac{dy}{dx}, \quad \text{where } n \text{ is an integer.}$$

Example 10.17

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$.

Solution

We differentiate both sides of the equation,

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for the derivative yields

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 10.18

Find the slopes of the tangent lines to the graph of $x^2 + y^2 = 4$ at the points corresponding to $x = 1$.

Solution

Substituting $x = 1$ into the given equation yields $y^2 = 3$ or $y = \pm\sqrt{3}$.

Hence, there are tangent lines at $\left(1, \sqrt{3}\right)$ and $\left(1, -\sqrt{3}\right)$. Although $\left(1, \sqrt{3}\right)$ and $\left(1, -\sqrt{3}\right)$ are points on the graphs of two different implicit functions, we got the correct slope of each point. We have

$$\frac{dy}{dx}\, at \left(1, \sqrt{3}\right) = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \frac{dy}{dx} \quad \text{at} \quad \left(1, -\sqrt{3}\right) = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Example 10.19

Find $\frac{dy}{dx}$ if $x^4 + x^2y^3 - y^5 = 2x + 1$.

Solution

Differentiating implicitly, we have

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(x^2y^3) - \frac{d}{dx}(y^5) = \frac{d}{dx}(2x + 1)$$

$$\text{This implies, } 4x^3 + x^2\left(3y^2\frac{dy}{dx}\right) + (2x)y^3 - 5y^4\frac{dy}{dx} = 2 + 0$$

$$\text{This implies, } 4x^3 + 3x^2y^2\frac{dy}{dx} + 2xy^3 - 5y^4\frac{dy}{dx} = 2$$

$$\text{This implies, } (3x^2y^2 - 5y^4)\frac{dy}{dx} = 2 - 4x^3 - 2xy^3$$

$$\text{or } \frac{dy}{dx} = \frac{2 - 4x^3 - 2xy^3}{3x^2y^2 - 5y^4}.$$

Example 10.20

Find $\frac{dy}{dx}$ if $\sin y = y\cos 2x$.

Solution

We have $\sin y = y\cos 2x$.

$$\text{Differentiating, } \frac{d}{dx}\sin y = \frac{d}{dx}(y\cos 2x)$$

$$\text{That is, } \cos y\frac{dy}{dx} = y(-2\sin 2x) + \cos 2x\frac{dy}{dx}$$

$$\text{This implies, } (\cos y - \cos 2x)\frac{dy}{dx} = -2y\sin 2x$$

$$\text{or } \frac{dy}{dx} = \frac{-2y\sin 2x}{\cos y - \cos 2x}.$$

10.4.4 Logarithmic Differentiation

By using the rules for differentiation and the table of derivatives of the basic elementary functions, we can now find automatically the derivatives of any elementary function, except for one type, the simplest representative of which is the function $y = x^x$. Such functions are described as power-exponential and include, in general, any function written as a power whose base and index both depend on the independent variable.

In order to find by the general rules the derivative of the power-exponential function $y = x^x$, we take logarithms on both sides to get

$\log y = x \log x, x > 0$

Since this is an identity, the derivative of the left-hand side must be equal to the derivative of the right, we obtain by differentiating with respect to x (keeping in mind the fact that the left hand side is a function of function) :

$$\frac{1}{y}\frac{dy}{dx} = \log x + 1$$

$$\text{Hence } \frac{dy}{dx} = y(\log x + 1) = x^x(\log x + 1)$$

The operation consists of first taking the logarithm of the function $f(x)$ (to base e) then differentiating is called **logarithmic differentiation** and its result

$$\boxed{\frac{d}{dx}(\log f(x)) = \frac{f'(x)}{f(x)}}.$$

is called the logarithmic derivative of $f(x)$.

The advantage in this method is that the calculation of derivatives of complicated functions involving products, quotients or powers can often be simplified by taking logarithms.

Example 10.21

Find the derivative of $y = \sqrt{x^2+4}\ .\sin^2 x\ .2^x$

Solution

Taking logarithm on both sides and using the law of logarithm,

we have, $\log y = \frac{1}{2}\log(x^2+4) + 2\log(\sin x) + x\log(2).$

$$\text{This implies, } \frac{y'}{y} = \frac{1}{2}\frac{2x}{x^2+4} + 2.\frac{\cos x}{\sin x} + \log 2$$

$$= \frac{x}{x^2+4} + 2\cot x + \log 2$$

$$\text{Therefore, } y' = \frac{dy}{dx} = y\left(\frac{x}{x^2+4} + 2\cot x + \log 2\right).$$

Example 10.22

Differentiate : $y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$.

Solution

Taking logarithm on both sides of the equation and using the rules of logarithm we have,

$\log y = \frac{3}{4}\log x + \frac{1}{2}\log(x^2+1) - 5\log(3x+2).$

Differentiating implicitly

$$\frac{y'}{y} = \frac{3}{4x} + \frac{1}{2}\frac{2x}{(x^2+1)} - \frac{5\times 3}{3x+2}$$

$$= \frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{3x+2}$$

$$\text{Therefore, } \frac{dy}{dx} = y' = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right].$$

Steps in Logarithmic Differentiation

(1) Take natural logarithm on both sides of an equation $y = f(x)$ and use the law of logarithms to simplify.

(2) Differentiate implicitly with respect to x.

(3) Solve the resulting equation for y'.

In general there are four cases for exponents and bases.

BCTJRW

(1) $\frac{d}{dx}(a^b) = 0$ (a and b are constants).

(2) $\frac{d}{dx}[f(x)]^b = b[f(x)]^{b-1} f'(x)$

(3) $\frac{d}{dx}[a^{g(x)}] = a^{g(x)}(\log a)g'(x)$

(4) $\frac{d}{dx}[f(x)]^{g(x)} = [f(x)]^{g(x)}\left[\frac{g(x)f'(x)}{f(x)} + \log f(x).g'(x)\right]$

Example 10.23

Differentiate $y = x^{\sqrt{x}}$.

Solution

Take logarithm :

$$\log y = \sqrt{x}\log x$$

Differentiating implicitly,

$$\frac{y'}{y} = \sqrt{x}.\frac{1}{x} + \frac{1}{2\sqrt{x}}.\log x$$

$$= \frac{\log x + 2}{2\sqrt{x}}$$

Therefore, $\frac{d}{dx}(x^{\sqrt{x}}) = y' = x^{\sqrt{x}}\left(\frac{\log x + 2}{2\sqrt{x}}\right)$.

10.4.5 Substitution method

It is very much useful in some processes of differentiation, in particular the differentiation involving inverse trigonometrical functions.

Consider $f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

For this function $f'(x)$ can be found out by using function of a function rule. But it is laborious. Instead we can use the substitution method.

$$\text{Take} \quad x = \tan\theta.$$

$$\text{Then} \quad \frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta \text{ and}$$

$$f(x) = \tan^{-1}(\tan 2\theta) = 2\theta$$

$$= 2\tan^{-1}x$$

$$f'(x) = \frac{2}{1+x^2}.$$

Example 10.24

If $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, find y'.

Solution

Let $x = \tan\theta$

$$\text{Then} \quad \frac{1+x}{1-x} = \frac{1+\tan\theta}{1-\tan\theta} = \tan\left(\frac{\pi}{4}+\theta\right).$$

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right] = \frac{\pi}{4}+\theta = \frac{\pi}{4}+\tan^{-1}x$$

$$y = \frac{\pi}{4}+\tan^{-1}x$$

$$y' = \frac{1}{1+x^2}.$$

Example 10.25

Find $f'(x)$ if $f(x) = \cos^{-1}(4x^3-3x)$.

Solution

$$\text{Let} \quad x = \cos\theta.$$

$$\text{Then} \quad 4x^3-3x = 4\cos^3\theta - 3\cos\theta = \cos 3\theta \text{ and}$$

$$f(x) = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x$$

$$\text{Therefore,} \quad f'(x) = 3\left(\frac{-1}{\sqrt{1-x^2}}\right) = \frac{-3}{\sqrt{1-x^2}}.$$

10.4.6 Derivatives of variables defined by parametric equations

Consider the equations $x = f(t)$, $y = g(t)$.

These equations give a functional relationship between the variables x and y. Given the value of t in some domain $[a, b]$, we can find x and y.

If two variables x and y are defined separately as a function of an intermediating (auxiliary) variable t, then the specification of a functional relationship between x and y is described as **parametric** and the auxiliary variable is known as **parameter**.

The operation of finding the direct connection between x and y without the presence of the auxiliary variable t is called **elimination of the parameter**. The question as to why should we deal with parametric equations is that two or more variables are reduced to a single variable, t.

For example, the equation of a circle with centre (0, 0) and radius r is $x^2 + y^2 = r^2$ This equation describes the relationship between x and y and the equations

$x = r\cos t$; $y = r\sin t$ are parametric equations.

Conversely, if we eliminate t, we get $x^2 + y^2 = r^2$

If y is regarded as a function of x then $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$.

If we regard x as a function of y, then the derivative of x with respect to y is,

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{f'(t)}{g'(t)}.$$

In the case of the circle, then $\dfrac{dy}{dx}$ is the slope of the tangent to the circle namely

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{r\cos t}{-r\sin t} = -\cot t \; .$$

Example 10.26

Find $\dfrac{dy}{dx}$ if $x = at^2$; $y = 2at$, $t \neq 0$.

Solution

We have $x = at^2$; $y = 2at$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2a}{2at} = \frac{1}{t}.$$

Example 10.27

Find $\dfrac{dy}{dx}$ if $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Solution

We have $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

Now $\dfrac{dx}{dt} = a(1 - \cos t)$; $\dfrac{dy}{dt} = a\sin t$

Therefore, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\sin t}{a(1-\cos t)} = \frac{\sin t}{(1-\cos t)} \; .$$

10.4.7 Differentiation of one function with respect to another function :

If $y = f(x)$ is differentiable, then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

If f and g are differentiable functions of x and if $\frac{dg}{dx} = g'(x) \neq 0$, then

$$\frac{df}{dg} = \frac{\frac{df}{dx}}{\frac{dg}{dx}} = \frac{f'(x)}{g'(x)}.$$

Example 10.28

Find the derivative of x^x with respect to $x \log x$.

Solution

Take $u = x^x, v = x\log x$

$$\log u = x\log x$$

$$\frac{1}{u}\frac{du}{dx} = x.\frac{1}{x} + 1.\log x = 1+\log x$$

$$\frac{du}{dx} = u(1+\log x) = x^x(1+\log x)$$

$$\frac{dv}{dx} = 1+\log x$$

$$\frac{d(x^x)}{d(x\log x)} = \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = x^x.$$

Note that when g is the identity function $g(x) = x$ then $\frac{df}{dg}$ reduces to $\frac{df}{dx} = f'(x)$.

Example 10.29

Find the derivative of $\tan^{-1}(1+x^2)$ with respect to x^2+x+1.

Solution :

Let
$$f(x) = \tan^{-1}(1+x^2)$$

$$g(x) = x^2+x+1$$

$$\frac{df}{dg} = \frac{f'(x)}{g'(x)}$$

$$f'(x) = \frac{2x}{\left(x^4+2x^2+2\right)}$$

$$g'(x) = 2x+1$$

$$\frac{df}{dg} = \frac{2x}{(2x+1)(x^4+2x^2+2)}.$$

Example 10.30

Differentiate $\sin(ax^2+bx+c)$ with respect to $\cos(lx^2+mx+n)$

Solution

Let $u = \sin(ax^2+bx+c)$

$$v = \cos(lx^2+mx+n)$$

$$\frac{du}{dv} = \frac{u'(x)}{v'(x)}$$

$$u'(x) = \cos(ax^2+bx+c)(2ax+b)$$

$$v'(x) = -\sin(lx^2+mx+n)(2lx+m)$$

$$\frac{du}{dv} = \frac{u'(x)}{v'(x)} = \frac{(2ax+b)\cos(ax^2+bx+c)}{-(2lx+m)\sin(lx^2+mx+n)}.$$

10.4.8 Higher order Derivatives

If $s = s(t)$ is the position function (displacement) of an object that moves in a straight line, we know that its first derivative has the simple physical interpretation as the velocity $v(t)$ of the object as a function of time :

$$v(t) = s'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{ds}{dt}.$$

The instantaneous rate of change of velocity with respect to time is called the acceleration $a(t)$ of the object. Then, the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

$$= \frac{d}{dt}(v(t))$$

$$= \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2} = s''(t)$$

Thus, if f is a differentiable function of x, then its first derivative $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ has a very simple geometrical interpretation as the slope of a tangent to the graph of $y = f(x)$. Since f' is also a function of x, f' may have a derivative of its own, and if it exists, denoted by $(f')' = f''$ is,

$$f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}.$$

$$= \frac{d}{dx}\left(f'(x)\right) = \frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$$

$$= \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2}$$

Other notations are $D^2 f(x) = D^2 y = y_2 = y''$

We can interpret a second derivative as a rate of change of a rate of change. But its geometrical interpretation is not so simple. However, there is a close connection exists between the second derivative $f''(x)$ and the radius of curvature of the graph of $y = f(x)$ which you will learn in higher classes.

Similarly, if f'' exists, it might or might not be differentiable. If it is, then the derivative of f'' is called **third derivative** of f and is denoted by

$$f'''(x) = \frac{d^3 y}{dx^3} = y''' = y_3.$$

We can interpret the third derivative physically in case when the function is the position function $f(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'(t)$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3 s}{dt^3}.$$

Thus, jerk is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

Example 10.31

Find y', y'' and y''' if $y = x^3 - 6x^2 - 5x + 3$.

Solution

We have, $y = x^3 - 6x^2 - 5x + 3$ and

$$y' = 3x^2 - 12x - 5$$

$$y'' = 6x - 12$$

$$y''' = 6.$$

Example 10.32

Find y''' if $y = \frac{1}{x}$.

Solution

We have, $y = \frac{1}{x} = x^{-1}$

$$y' = -1x^{-2} = -\frac{1}{x^2}$$

$$y'' = (-1)(-2)x^{-3} = \frac{(-1)^2 2!}{x^3}.$$

$$\text{and} \quad y''' = (-1)(-2)(-3)x^{-4} = \frac{(-1)^3 3!}{x^4}.$$

Example 10.33

Find f'' if $f(x) = x \cos x$.

Solution

$$\text{We have, } f(x) = x \cos x.$$

$$\text{Now } f'(x) = -x \sin x + \cos x, \text{ and}$$

$$f''(x) = -(x \cos x + \sin x) - \sin x$$

$$= -x \cos x - 2 \sin x.$$

Example 10.34

Find y'' if $x^4 + y^4 = 16$.

Solution

We have $x^4 + y^4 = 16$.

Differentiating implicitly, $4x^3 + 4y^3 y' = 0$

Solving for y' gives

$$y' = -\frac{x^3}{y^3}.$$

To find y'' we differentiate this expression for y' using the quotient rule and remembering that y is a function of x.

$$y'' = \frac{d}{dx}\left(\frac{-x^3}{y^3}\right) = \frac{-\left[y^3 \frac{d}{dx}(x^3) - x^3 \frac{d}{dx}(y^3)\right]}{(y^3)^2}$$

$$= -\frac{\left[y^3.3x^2 - x^3(3y^2 y')\right]}{y^6}$$

$$= -\frac{3x^2 y^3 - 3x^3 y^2\left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= -\frac{3(x^2 y^4 + x^6)}{y^7} = \frac{-3x^2[x^4 + y^4]}{y^7}$$

$$= \frac{-3x^2(16)}{y^7} = \frac{-48x^2}{y^7}.$$

Example 10.35

Find the second order derivative if x and y are given by

$$x = a \cos t$$

$$y = a \sin t.$$

Solution

Differentiating the function implicitly with respect to x, we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{-a\sin t} = -\frac{\cos t}{\sin t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{-\cos t}{\sin t}\right)$$

$$= \frac{d}{dt}\left(\frac{-\cos t}{\sin t}\right)\frac{dt}{dx} = -[-\text{cosec}^2 t]\times\frac{1}{x'(t)}$$

$$= \text{cosec}^2 t\times\frac{1}{-a\sin t}$$

$$= -\frac{\cos ec^3 t}{a}.$$

Example 10.36

Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=4$.

Solution

We have $\quad x^2+y^2 = 4$

As before, $\quad \frac{dy}{dx} = -\frac{x}{y}$

Hence, by the quotient rule

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{y.1-x.\frac{dy}{dx}}{y^2}$$

$$= -\frac{y-x\left(-\frac{x}{y}\right)}{y^2}$$

$$= -\frac{x^2+y^2}{y^3} = -\frac{4}{y^3}.$$

EXERCISE 10.4

Find the derivatives of the following (1 – 18) :

(1) $y = x^{\cos x}$

(2) $y = x^{\log x} + (\log x)^x$

(3) $\sqrt{xy} = e^{(x-y)}$

(4) $x^y = y^x$

(5) $(\cos x)^{\log x}$

(6) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(7) $\sqrt{x^2 + y^2} = \tan^{-1}\left(\dfrac{y}{x}\right)$

(8) $\tan(x + y) + \tan(x - y) = x$

(9) If $\cos(xy) = x$, show that $\dfrac{dy}{dx} = \dfrac{-(1 + y\sin(xy))}{x\sin xy}$

(10) $\tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}$

(11) $\tan^{-1}\left(\dfrac{6x}{1-9x^2}\right)$

(12) $\cos\left(2\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}\right)$

(13) $x = a\cos^3 t$; $y = a\sin^3 t$

(14) $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

(15) $x = \dfrac{1-t^2}{1+t^2}, y = \dfrac{2t}{1+t^2}$

(16) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

(17) $\sin^{-1}(3x - 4x^3)$

(18) $\tan^{-1}\left(\dfrac{\cos x + \sin x}{\cos x - \sin x}\right)$.

(19) Find the derivative of $\sin x^2$ with respect to x^2.

(20) Find the derivative of $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ with respect to $\tan^{-1} x$.

(21) If $u = \tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$ and $v = \tan^{-1} x$, find $\dfrac{du}{dv}$.

(22) Find the derivative with $\tan^{-1}\left(\dfrac{\sin x}{1+\cos x}\right)$ with respect to $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$.

(23) If $y = \sin^{-1} x$ then find y''.

(24) If $y = e^{\tan^{-1} x}$, show that $(1+x^2)y'' + (2x-1)y' = 0$.

(25) If $y = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, show that $(1-x^2)y_2 - 3xy_1 - y = 0$.

(26) If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ then prove that at $\theta = \dfrac{\pi}{2}, y'' = \dfrac{1}{a}$.

(27) If $\sin y = x\sin(a+y)$, then prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$, $a \neq n\pi$.

(28) If $y = (\cos^{-1} x)^2$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$. Hence find y_2 when $x = 0$

EXERCISE 10.5

BD3FTJ

Choose the correct or the most suitable answer from the given four alternatives.

(1) $\frac{d}{dx}\left(\frac{2}{\pi}\sin x^\circ\right)$ is

(1) $\frac{\pi}{180}\cos x^\circ$ (2) $\frac{1}{90}\cos x^\circ$ (3) $\frac{\pi}{90}\cos x^\circ$ (4) $\frac{2}{\pi}\cos x^\circ$

(2) If $y = f(x^2+2)$ and $f'(3) = 5$, then $\frac{dy}{dx}$ at $x = 1$ is

(1) 5 (2) 25 (3) 15 (4) 10

(3) If $y = \frac{1}{4}u^4$, $u = \frac{2}{3}x^3 + 5$, then $\frac{dy}{dx}$ is

(1) $\frac{1}{27}x^2(2x^3+15)^3$ (2) $\frac{2}{27}x(2x^3+5)^3$

(3) $\frac{2}{27}x^2(2x^3+15)^3$ (4) $-\frac{2}{27}x(2x^3+5)^3$

(4) If $f(x) = x^2 - 3x$, then the points at which $f(x) = f'(x)$ are

(1) both positive integers (2) both negative integers

(3) both irrational (4) one rational and another irrational

(5) If $y = \frac{1}{a-z}$, then $\frac{dz}{dy}$ is

(1) $(a-z)^2$ (2) $-(z-a)^2$ (3) $(z+a)^2$ (4) $-(z+a)^2$

(6) If $y = \cos(\sin x^2)$, then $\frac{dy}{dx}$ at $x = \sqrt{\frac{\pi}{2}}$ is

(1) – 2 (2) 2 (3) $-2\sqrt{\frac{\pi}{2}}$ (4) 0

(7) If $y = mx + c$ and $f(0) = f'(0) = 1$, then $f(2)$ is

(1) 1 (2) 2 (3) 3 (4) – 3

(8) If $f(x) = x\tan^{-1}x$, then $f'(1)$ is

(1) $1+\frac{\pi}{4}$ (2) $\frac{1}{2}+\frac{\pi}{4}$ (3) $\frac{1}{2}-\frac{\pi}{4}$ (4) 2

(9) $\frac{d}{dx}(e^{x+5\log x})$ is

(1) $e^x.x^4(x+5)$ (2) $e^x.x(x+5)$ (3) $e^x+\frac{5}{x}$ (4) $e^x-\frac{5}{x}$

(10) If the derivative of $(ax-5)e^{3x}$ at $x = 0$ is – 13, then the value of a is

(1) 8 (2) – 2 (3) 5 (4) 2

(11) $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}$ then $\frac{dy}{dx}$ is

(1) $-\frac{y}{x}$ (2) $\frac{y}{x}$ (3) $-\frac{x}{y}$ (4) $\frac{x}{y}$

(12) If $x = a\sin\theta$ and $y = b\cos\theta$, then $\frac{d^2y}{dx^2}$ is

(1) $\frac{a}{b^2}\sec^2\theta$ (2) $-\frac{b}{a}\sec^2\theta$ (3) $-\frac{b}{a^2}\sec^3\theta$ (4) $-\frac{b^2}{a^2}\sec^3\theta$

(13) The differential coefficient of $\log_{10} x$ with respect to $\log_x 10$ is

(1) 1 (2) $-(\log_{10} x)^2$ (3) $(\log_x 10)^2$ ` (4) $\frac{x^2}{100}$

(14) If $f(x) = x + 2$, then $f'(f(x))$ at $x = 4$ is

(1) 8 (2) 1 (3) 4 (4) 5

(15) If $y = \frac{(1-x)^2}{x^2}$, then $\frac{dy}{dx}$ is

(1) $\frac{2}{x^2}+\frac{2}{x^3}$ (2) $-\frac{2}{x^2}+\frac{2}{x^3}$ (3) $-\frac{2}{x^2}-\frac{2}{x^3}$ (4) $-\frac{2}{x^3}+\frac{2}{x^2}$

(16) If $pv = 81$, then $\frac{dp}{dv}$ at $v = 9$ is

(1) 1 (2) – 1 (3) 2 (4) –2

(17) If $f(x) = \begin{cases} x-5 & \text{if } x \le 1 \\ 4x^2-9 & \text{if } 1 < x < 2 \\ 3x+4 & \text{if } x \ge 2 \end{cases}$, then the right hand derivative of $f(x)$ at $x = 2$ is

(1) 0 (2) 2 (3) 3 (4) 4

(18) It is given that $f'(a)$ exists, then $\lim_{x\to a} \frac{xf(a)-af(x)}{x-a}$ is

(1) $f(a)-af'(a)$ (2) $f'(a)$ (3) $-f'(a)$ (4) $f(a)+af'(a)$

(19) If $f(x) = \begin{cases} x+1, & \text{when } x < 2 \\ 2x-1 & \text{when } x \ge 2 \end{cases}$, then $f'(2)$ is

(1) 0 (2) 1 (3) 2 (4) does not exist

(20) If $g(x) = (x^2+2x+1)f(x)$ and $f(0) = 5$ and $\lim_{x\to 0}\frac{f(x)-5}{x}$=4, then $g'(0)$ is

(1) 20 (2) 14 (3) 18 (4) 12

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(21) If $f(x)=\begin{cases} x+2, & -1<x<3 \\ 5 & x=3 \\ 8-x & x>3 \end{cases}$, then at $x=3$, $f'(x)$ is

(1) 1 (2) – 1 (3) 0 (4) does not exist

(22) The derivative of $f(x)=x|x|$ at $x=-3$ is

(1) 6 (2) – 6 (3) does not exist (4) 0

(23) If $f(x)=\begin{cases} 2a-x, & \text{for } -a<x<a \\ 3x-2a & \text{for } x\geq a \end{cases}$, then which one of the following is true?

(1) $f(x)$ is not differentiable at $x = a$ (2) $f(x)$ is discontinuous at $x = a$

(3) $f(x)$ is continuous for all x in $\mathbb{R}$ (4) $f(x)$ is differentiable for all $x \geq a$

(24) If $f(x)=\begin{cases} ax^2-b, & -1<x<1 \\ \dfrac{1}{|x|}, & \text{elsewhere} \end{cases}$ is differentiable at $x = 1$, then

(1) $a=\frac{1}{2}$, $b=\frac{-3}{2}$ (2) $a=\frac{-1}{2}$, $b=\frac{3}{2}$ (3) $a=-\frac{1}{2}$, $b=-\frac{3}{2}$ (4) $a=\frac{1}{2}$, $b=\frac{3}{2}$

(25) The number of points in $\mathbb{R}$ in which the function $f(x)=|x-1|+|x-3|+\sin x$ is not differentiable, is

(1) 3 (2) 2 (3) 1 (4) 4

SUMMARY

In this chapter we have acquired the knowledge of

- Derivative as a rate of change. If $y=f(x)$ then the derivative of y with respect to x at x_0 is $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$, provided the limit exists, where existence means
$$f'(x_0^+)=\lim_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=\lim_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0^-)$$
is a unique real number.
- Derivative of $y=f(x)$ at $x=x_0$ is $\left(\dfrac{dy}{dx}\right)_{x=x_0}=f'(x_0)=\lim\limits_{x\to x_0}\dfrac{f(x)-f(x_0)}{x-x_0}$.
- Geometrical meaning of the derivative of $y=f(x)$ is the slope of the tangent to the curve $y=f(x)$ at $(x, f(x))$.
- Physical meaning of the derivative of $s=f(t)$ with respect to t is the rate of change of displacement, that is velocity. The second derivative is acceleration and the third derivative is jerk.
- Discontinuity of $y=f(x)$ at $x=x_0$ implies non-differentiability of $f(x)$ at $x=x_0$.

- Non-existence of the derivative of $y = f(x)$ at $x = x_0$ implies that the graph of $y = f(x)$ fails to admit tangent at $(x_0, f(x_0))$.
- Geometrically, if the graph of $y = f(x)$ admits cups $(\vee)$ or caps $(\wedge)$ at $x = x_0$ then derivative at $x = x_0$ does not exist.
- Derivative should be understood as a process not as a set of rules.
- Differentiability implies continuity, but the converse is not true.
- The sum, difference, product and composite of differentiable function is differentiable, and the quotient of two differentiable function is differentiable wherever it is defined.

(i) $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

(ii) $\frac{d}{dx}(f(x)g(x)) = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$

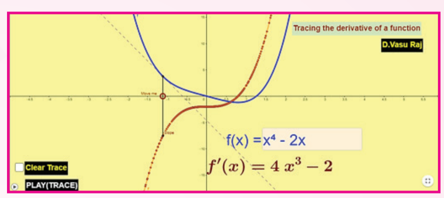
(iii) $\frac{d}{dx}((fog)(x)) = f'(g(x))g'(x)$

(iv) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$, where $g(x) \neq 0$.

Differential Calculus

ICT CORNER 10(a)

Expected Outcome



Step 1

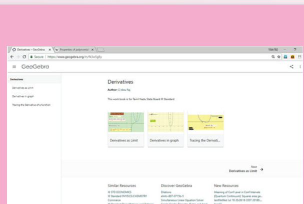
Open the Browser type the URL Link given below (or) Scan the QR Code.

GeoGebra Workbook called "Derivatives" will appear. In that there are several worksheets related to your lesson.

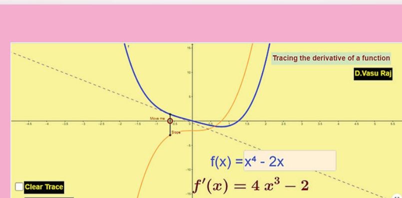
Step 2

Select the work sheet "Tracing the derivative of a function". You can enter any function in *f*(*x*) box. You can see the function in blue colour and derivative in orange colour. Click play trace button to get animation of the locus of derivative(*x*, slope at *x*)

Observe the trace and find that derivative is the path of slope at each point on *f*(*x*).



Step1



Step2

B162_11_MAT_EM

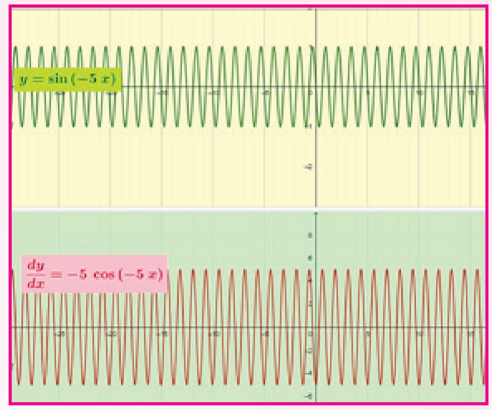
Browse in the link:

Derivatives: https://ggbm.at/fk3w5g8y

ICT CORNER 10(b)

Differential Calculus

Expected Outcome



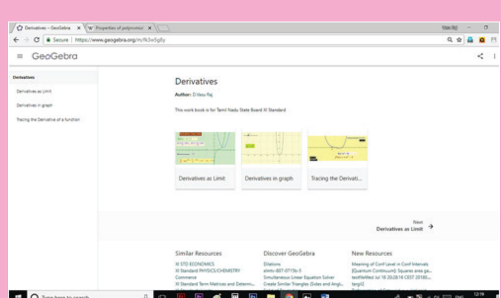
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "Derivatives" will appear. In that there are several worksheets related to your lesson.

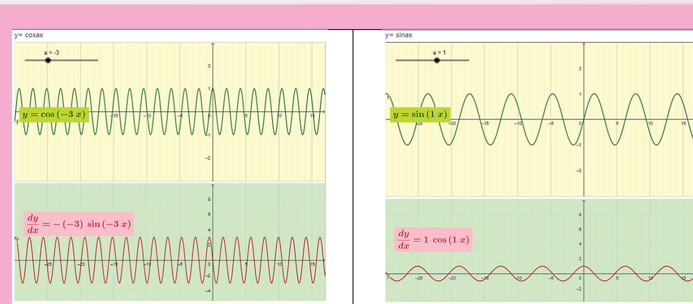
Step 2

Select the work sheet "Derivatives in graph". Some basic functions and their derivatives are given one under another

You can change "a" value by moving the slider and observe the changes in each function and its derivatives.



Step1



Step2

Browse in the link:

Derivatives: https://ggbm.at/fk3w5g8y

B162_11_MAT_EM

Chapter 11 Integral Calculus

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It is worth noting that the notation facilitates discovery.
This, in most wonderful way, reduces the minds labour.
- Gottfried Wilhelm Leibnitz

11.1 Introduction

Newton

Leibnitz

Gottfried Wilhelm Leibnitz (1646-1716) and Sir Isaac Newton (1643-1727) independently discovered calculus in the mid-17th century. **Leibnitz,** a German philosopher, mathematician, and political adviser, importantly both as a metaphysician and as a logician, was a distinguished independent inventor of the Differential and Integral Calculus. Sir Isaac Newton had created an expression for the area under a curve by considering a momentary increase at a point. In effect, the fundamental theorem of calculus was built into his calculations. His work and discoveries were not limited to mathematics; he also developed theories in optics and gravitation.

One cannot imagine a world without differentiation and integration. In this century, we witnessed remarkable scientific advancement owing to the ingenious application of these two basic components of Mathematics. Calculus serve as unavoidable tool for finding solutions to the variety of problems that arise in physics, astronomy, engineering, chemistry, geology, biology, and social sciences.

Calculus deals principally with two geometric problems.

(i) The problem of finding **SLOPE** of the tangent line to the curve, is studied by the limiting process known as **differentiation** and

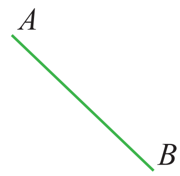
(ii) Problem of finding the **AREA** of a region under a curve is studied by another limiting process called **Integration**.

In chapters 9 and 10, we have studied the differential calculus. In this chapter let us study some fundamentals of integration.

Consider some simple situations illustrated below.

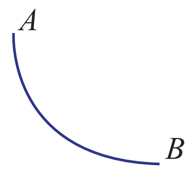
Situation 1

The shortest distance between two points A and B in a plane is the line segment joining the straight line A and B. Suppose it is required to find the line connecting two points A and B that do not lie on a vertical line such that a moving particle slides down on this line from A to B in the shortest time (minimum time). Most of us believe that the shortest distance path in Fig. 11.1(a) will take the shortest time.



The shortest distance path between A and B

Fig. 11.1 (a)



The shortest time route between A and B

Fig. 11.1 (b)

Certainly this route is not the shortest time route joining the points A and B, because the velocity of the motion in the straight line (Fig. 11.1(a)) will be comparatively slow; whereas it take a curve that is steeper near A (Fig. 11.1(b)), even though the path becomes longer, a considerable portion of the distance will be covered at a greater speed. The solution to this problem is solved by **Integral calculus.** This is called Brachistochrone problem which initiates the study of calculus of variation using integral tool.

Situation 2

In elementary geometry we have learnt to evaluate the measure of the following regular shape of the figures given below by using known formulae.

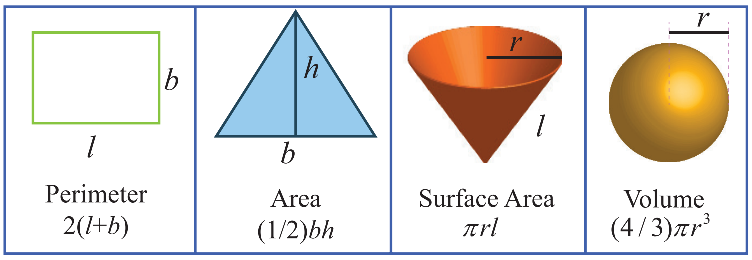


Fig. 11.2 (a)

How can the measure of the following figures given by functions be calculated?

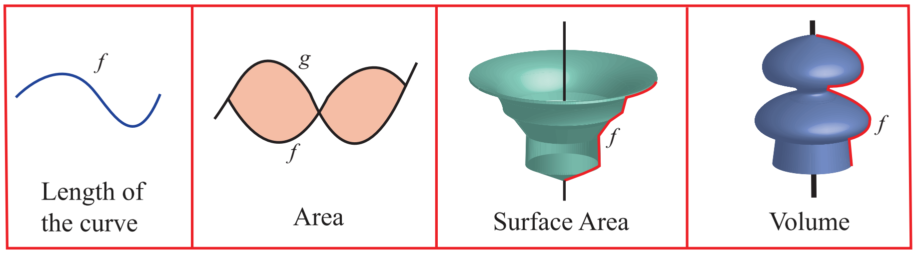


Fig. 11.2 (b)

Though the problems look so difficult, integral calculus solves it without any difficulties.

Situation 3

At a particular moment, a student needs to stop his speedy bike to avoid a collision with the barrier ahead at a distance 40 metres away from him. Immediately he slows (acceleration) down the bike applying brake at a rate of 8 meter/second^2. If the bike is moving at a speed of 24m/s, when the brakes are applied, will it stop before collision?

Fig. 11.3

Also look at the following problems that occur naturally in our life.

- What speed has to be applied to fire a satellite upward so that it never returns to the earth?
- What is the radius of the smallest circular disk that can cover every isosceles triangle of given perimeter P?
- What volume of material is removed from a solid sphere of radius $2r$ if a hole of radius r is drilled through the centre?
- If a strain of bacteria grows at a rate proportional to the amount present and if the population doubles in one hour, how much will it increase at the end of two hours?

Integration will answer for all the above problems.

Learning Objectives

On completion of this chapter, the students are expected to

- understand the definition of an indefinite integral as a result of reversing the process of differentiation
- find the indefinite integrals of sums, differences and constant multiples of certain elementary functions.
- use the appropriate techniques to find the indefinite integrals of composite functions.
- apply integration to find the function, when the rate of change of function is given.

11.2 Newton-Leibnitz Integral

Integral calculus is mainly divided into indefinite integrals and definite integrals. In this chapter, we study indefinite integration, the process of obtaining a function from its derivative.

We are already familiar with inverse operations. $(+,-)$, $(\times, \div)$, $\left((\ )^n, \sqrt[n]{\ }\right)$ are some pairs of inverse operations. Similarly differentiation and integrations $(d, \int)$ are also inverse operations. In this section we develop the inverse operation of differentiation called 'antidifferentiation'.

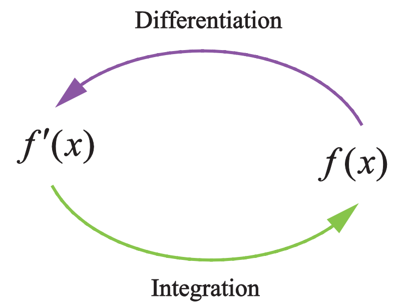


Fig. 11.4

Definition 11.1

A function $F(x)$ is called an **antiderivative** (Newton-Leibnitz integral or primitive) of a function $f(x)$ on an interval I if

$$F'(x) = f(x), \text{ for every value of } x \text{ in } I$$

Illustration 11.1

If $F(x) = x^2 + 5$ then

$F'(x) = 2x$.

Thus if $f(x)$ is defined by

$f(x) = 2x$, then

we say that $f(x)$ is the derivative of $F(x)$ and that $F(x)$ is an antiderivative of $f(x)$

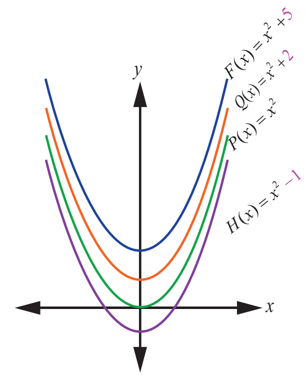


Fig. 11.5

Consider the following table

$F(x)$	$F'(x) = f(x)$	Antiderivative of $f(x) = 2x$
$P(x) = x^2 + 0$ $Q(x) = x^2 + 2$ $H(x) = x^2 - 1$	$\left.\begin{array}{l} P'(x) = 2x \\ Q'(x) = 2x \\ H'(x) = 2x \end{array}\right\} f(x) = 2x$	$F(x) = x^2 + ?$

We can see that the derivative of $F(x), P(x), Q(x)$ and $H(x)$ is $f(x)$, but in reverse the antiderivatives of $f(x) = 2x$ is not unique. That is the antiderivatives of $f(x)$ is a family of infinitely many functions.

Theorem 11.1

If $F(x)$ is a particular antiderivative of a function $f(x)$ on an interval *I*, then every antiderivative of $f(x)$ on *I* is given by

$$\int f(x)dx = F(x) + c$$

where c is called an arbitrary constant, and all antiderivatives of $f(x)$ on *I* can be obtained by assigning particular value to c.

The function ***f(x)*** is called **Integrand**.

The variable ***x*** in **d*x*** is called **variable of integration** or **integrator**.

The process of finding the integral is called **integration** or **antidifferentiation** (Newton-Leibnitz integral).

The peculiar integral sign $\int$ originates in an elongated S (like Σ) which stands for sum.

Often in applications involving differentiation it is desired to find a particular integral antiderivative that satisfies certain conditions called initial condition or boundary conditions.

For instance, if an equation involving $\frac{dy}{dx}$ is given as well as the initial condition that $y = y_1$ when $x = x_1$, then after the set of all antiderivatives is found, if x and y are replaced by x_1 and y_1, a particular value of the arbitrary constant is determined. With this value of c a particular antiderivative is obtained.

Illustration 11.2

Suppose we wish to find the particular antiderivative satisfying the equation

$$\frac{dy}{dx} = 2x$$

and the initial condition that $y = 10$ when $x = 2$.
From the given equation

$$\frac{dy}{dx} = 2x$$
$$y = \int 2x\,dx$$
$$y = x^2 + c$$

We substitute $y = 10$ when $x = 2$, in the above equation

$$10 = 2^2 + c \Rightarrow c = 6$$

When this value c =6 is substituted we obtain

$$y = x^2 + 6$$

which gives the particular antiderivative desired.

11.3 Basic Rules of Integration

Standard results:

Since integration is the reverse process of differentiation, the basic integration formulae given below can be derived directly from their corresponding derivative formulae from earlier chapter.

Derivatives	Antiderivatives
$\frac{d}{dx}(c) = 0,$ where c is a constant	$\int 0\,dx = c$, where c is a constant
$\frac{d}{dx}(kx) = k,$ where k is a constant	$\int k\,dx = kx + c$ where c is a constant
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$ (Power rule)
$\frac{d}{dx}(\log x) = \left(\frac{1}{x}\right)$	$\int \frac{1}{x}\,dx = \log\lvert x\rvert + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$

$\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$	$\int \text{cosec}^2 x \, dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x \, dx = \sec x + c$
$\frac{d}{dx}(-\text{cosec} x) = \text{cosec} x \cot x$	$\int \text{cosec} x \cot x \, dx = -\text{cosec} x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x dx = e^x + c$
$\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$	$\int a^x dx = \frac{a^x}{\log a} + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

Example 11.1

Integrate the following with respect to x.

(i) x^{10} (ii) $\frac{1}{x^{10}}$ (iii) $\sqrt{x}$ (iv) $\frac{1}{\sqrt{x}}$

Solution

(i) We know that $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1.$

Putting $n = 10$, we get

$$\int x^{10} dx = \frac{x^{10+1}}{10+1} + c = \frac{x^{11}}{11} + c$$

(ii) $$\int \frac{1}{x^{10}} dx = \int x^{-10} dx = \frac{x^{-10+1}}{-10+1} + c = -\frac{1}{9x^9} + c$$

(iii) $$\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} x^{\frac{3}{2}} + c$$

(iv) $$\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = 2\sqrt{x} + c$$

Example 11.2

Integrate the following with respect to x.

(i) $\dfrac{1}{\cos^2 x}$ (ii) $\dfrac{\cot x}{\sin x}$ (iii) $\dfrac{\sin x}{\cos^2 x}$ (iv) $\dfrac{1}{\sqrt{1-x^2}}$

Solution

(i) $\displaystyle\int \frac{1}{\cos^2 x}dx = \int \sec^2 x\,dx = \tan x + c$

(ii) $\displaystyle\int \frac{\cot x}{\sin x}dx = \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + c$

(iii) $\displaystyle\int \frac{\sin x}{\cos^2 x}dx = \int \frac{\sin x}{\cos x}.\frac{1}{\cos x}dx = \int \tan x \sec x\,dx = \sec x + c$

(iv) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + c$

Example 11.3

Integrate the following with respect to x:

(i) $\dfrac{1}{e^{-x}}$ (ii) $\dfrac{x^2}{x^3}$ (iii) $\dfrac{1}{x^3}$ (iv) $\dfrac{1}{1+x^2}$

Solution

(i) $\displaystyle\int \frac{1}{e^{-x}}\,dx = \int e^x dx = e^x + c$

(ii) $\displaystyle\int \frac{x^2}{x^3}dx = \int \frac{1}{x}dx = \log|x| + c$

(iii) $\displaystyle\int \frac{1}{x^3}dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + c = -\frac{1}{2x^2} + c$

(iv) $\displaystyle\int \frac{1}{1+x^2}dx = \tan^{-1} x + c$

EXERCISE 11.1

Integrate the following with respect to x:

(1) (i) x^{11} (ii) $\dfrac{1}{x^7}$ (iii) $\sqrt[3]{x^4}$ (iv) $(x^5)^{\frac{1}{8}}$

(2) (i) $\dfrac{1}{\sin^2 x}$ (ii) $\dfrac{\tan x}{\cos x}$ (iii) $\dfrac{\cos x}{\sin^2 x}$ (iv) $\dfrac{1}{\cos^2 x}$

(3) (i) 12^3 (ii) $\dfrac{x^{24}}{x^{25}}$ (iii) e^x

(4) (i) $(1+x^2)^{-1}$ (ii) $\left(1-x^2\right)^{-\frac{1}{2}}$

11.4 Integrals of the Form $\int f(ax+b)dx$

We know that

$$\frac{d}{dx}\left[\frac{(x-a)^{10}}{10}\right] = (x-a)^9 \Rightarrow \int (x-a)^9\,dx = \frac{(x-a)^{10}}{10} + c$$

$$\frac{d}{dx}[\sin(x+k)] = \cos(x+k) \Rightarrow \int \cos(x+k)dx = \sin(x+k) + c$$

It is clear that whenever a constant is added or subtracted with the independent variable x, the fundamental formulae remain the same.

But

$$\frac{d}{dx}\left[\frac{1}{l}(e^{lx+m})\right] = e^{lx+m} \Rightarrow \int e^{lx+m}dx = \frac{1}{l}e^{(lx+m)} + c$$

$$\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)\right] = \cos(ax+b) \Rightarrow \int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b)+c$$

Here, if any constant is multiplied with the independent variable x, then the same fundamental formula can be used after dividing it by the coefficient of x

That is, $\boxed{\text{if } \int f(x)dx = g(x)+c, \text{ then } \int f(ax+b)dx = \frac{1}{a}g(ax+b)+c}$

The above formula can also be derived by using substitution method, which will be studied later.

Example 11.4

Evaluate the following with respect to x:

(i) $\int (4x+5)^6\, dx$ (ii) $\int \sqrt{(15-2x)}\, dx$ (iii) $\int \frac{1}{(3x+7)^4} dx$

Solution

(i) $\int (4x+5)^6\, dx = \frac{1}{4}\frac{(4x+5)^{6+1}}{6+1} = \frac{(4x+5)^7}{28}+c$

(ii) $\int \sqrt{(15-2x)}\, dx = \int (15-2x)^{\frac{1}{2}}\, dx = \left(\frac{1}{-2}\right)\frac{(15-2x)^{\frac{1}{2}+1}}{(1/2)+1} = -\frac{(15-2x)^{\frac{3}{2}}}{3}+c$

(iii) $\int \frac{1}{(3x+7)^4}dx = \int (3x+7)^{-4}\, dx = \frac{1}{3}\frac{(3x+7)^{-4+1}}{-4+1} = -\frac{1}{9(3x+7)^3}+c$

Example 11.5

Integrate the following with respect to x:

(i) $\sin(2x + 4)$ (ii) $\sec^2(3+4x)$ (iii) $\operatorname{cosec}(ax+b)\cot(ax+b)$

Solution

(i) $\int \sin(2x + 4)\, dx = \left(\frac{1}{2}\right)(-\cos(2x+4))+c = -\frac{1}{2}\cos(2x+4)+c$

(ii) $\int \sec^2(3+4x)\, dx = \frac{1}{4}\tan(3+4x)+c$

(iii) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\, dx = \left(\frac{1}{a}\right)(-\operatorname{cosec}(ax+b))+c = -\frac{1}{a}\operatorname{cosec}(ax+b)+c$

Example 11.6

Integrate the following with respect to x:

(i) e^{3x} (ii) e^{5-4x} (iii) $\frac{1}{(3x-2)}$ (iv) $\frac{1}{(5-4x)}$

Solution

(i) $\int e^{3x}\, dx = \frac{1}{3}e^{3x}+c$ (ii) $\int e^{5-4x}\, dx = -\frac{e^{5-4x}}{4}+c$

(iii) $\int \frac{1}{(3x-2)}dx = \frac{1}{3}\log|(3x-2)|+c$ (iv) $\int \frac{1}{(5-4x)}dx = -\frac{1}{4}\log|(5-4x)|+c$

Example 11.7

Integrate the following with respect to x:

(i) $\frac{1}{1+(2x)^2}$ (ii) $\frac{1}{\sqrt{1-(9x)^2}}$ (iii) $\frac{1}{\sqrt{1-25x^2}}$

Solution

(i) $\int \frac{1}{1+(2x)^2}dx = \frac{1}{2}\tan^{-1}(2x)+c$ (ii) $\int \frac{1}{\sqrt{1-(9x)^2}}dx = \frac{1}{9}\sin^{-1}(9x)+c$

(iii) $\int \frac{1}{\sqrt{1-25x^2}}dx = \int \frac{1}{\sqrt{1-(5x)^2}}dx = \frac{1}{5}\sin^{-1}(5x)+c$

EXERCISE 11.2

Integrate the following functions with respect to x:

(1) (i) $(x+5)^6$ (ii) $\frac{1}{(2-3x)^4}$ (iii) $\sqrt{3x+2}$

(2) (i) $\sin 3x$ (ii) $\cos(5-11x)$ (iii) $\text{cosec}^2(5x-7)$

(3) (i) e^{3x-6} (ii) e^{8-7x} (iii) $\frac{1}{6-4x}$

(4) (i) $\sec^2\frac{x}{5}$ (ii) $\text{cosec}(5x+3)\cot(5x+3)$ (iii) $\sec(2-15x)\tan(2-15x)$

(5) (i) $\frac{1}{\sqrt{1-(4x)^2}}$ (ii) $\frac{1}{\sqrt{1-81x^2}}$ (iii) $\frac{1}{1+36x^2}$

11.5 Properties of Integrals

(1) If k is any constant, then $\int kf(x)dx = k\int f(x)dx$

(2) $\int (f_1(x) \pm f_2(x))dx = \int f_1(x)dx \pm \int f_2(x)dx$

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Note 11.1

The above two properties can be combined and extended as

$$\int (k_1f_1(x) \pm k_2f_2(x) \pm k_3f_3(x) \pm \cdots \pm k_nf_n(x))\,dx$$
$$= k_1\int f_1(x)dx \pm k_2\int f_2(x)dx \pm k_3\int f_3(x)dx \pm \cdots \pm k_n\int f_n(x)dx.$$

That is, the integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals

Example 11.8

Integrate the following with respect to x:

(i) $5x^4$ (ii) $5x^2-4+\frac{7}{x}+\frac{2}{\sqrt{x}}$ (iii) $2\cos x-4\sin x+5\sec^2 x+\text{cosec}^2 x$

Solution

(i) $\int 5x^4 dx \quad = \quad 5\int x^4 dx = 5\dfrac{x^{4+1}}{4+1} = 5\dfrac{x^5}{5} = x^5 + c.$

(ii) $\displaystyle\int\left(5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\right)dx \quad = \quad 5\int x^2 dx - 4\int dx + 7\int\frac{1}{x}dx + 2\int\frac{1}{\sqrt{x}}dx$

$$= 5\frac{x^{2+1}}{2+1} - 4x + 7\log|x| + 2\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$

$$= \frac{5}{3}x^3 - 4x + 7\log|x| + 4\sqrt{x} + c$$

(iii) $\int(2\cos x - 4\sin x + 5\sec^2 x + \operatorname{cosec}^2 x)dx$

$$= 2\int\cos x\,dx - 4\int\sin x\,dx + 5\int\sec^2 x\,dx + \int\operatorname{cosec}^2 x\,dx$$

$$= 2\sin x + 4\cos x + 5\tan x - \cot x + c$$

Example 11.9

Evaluate the following integrals:

(i) $\dfrac{12}{(4x-5)^3} + \dfrac{6}{3x+2} + 16e^{4x+3}$ (ii) $\dfrac{15}{\sqrt{5x-4}} - 8\cot(4x+2)\operatorname{cosec}(4x+2)$

Solution

(i) $\displaystyle\int\left(\frac{12}{(4x-5)^3} + \frac{6}{3x+2} + 16e^{4x+3}\right)dx$

$$= 12\int\frac{1}{(4x-5)^3}dx + 6\int\frac{1}{3x+2}dx + 16\int e^{4x+3}dx$$

$$= 12\left(\frac{1}{4}\right)\left(-\frac{1}{2(4x-5)^2}\right) + 6\left(\frac{1}{3}\right)\log|3x+2| + 16\left(\frac{1}{4}\right)e^{4x+3} + c$$

$$= -\frac{3}{2(4x-5)^2} + 2\log|3x+2| + 4e^{4x+3} + c.$$

(ii) $\displaystyle\int\left(\frac{15}{\sqrt{5x-4}} - 8\cot(4x+2)\operatorname{cosec}(4x+2)\right)dx$

$$= 15\int\frac{1}{\sqrt{5x-4}}dx - 8\int\cot(4x+2)\operatorname{cosec}(4x+2)\,dx$$

$$= 15\left(\frac{1}{5}\right)(2\sqrt{5x-4}) - 8\left(\frac{1}{4}\right)(-\operatorname{cosec}(4x+2)) + c$$

$$= 6\sqrt{5x-4} + 2\operatorname{cosec}(4x+2) + c.$$

EXERCISE 11.3

Integrate the following with respect to x:

(1) $(x+4)^5+\frac{5}{(2-5x)^4}-\operatorname{cosec}^2(3x-1)$

(2) $4\cos(5-2x)+9e^{3x-6}+\frac{24}{6-4x}$

(3) $\sec^2\frac{x}{5}+18\cos 2x+10\sec(5x+3)\tan(5x+3)$

(4) $\frac{8}{\sqrt{1-(4x)^2}}+\frac{27}{\sqrt{1-9x^2}}-\frac{15}{1+25x^2}$

(5) $\frac{6}{1+(3x+2)^2}-\frac{12}{\sqrt{1-(3-4x)^2}}$

(6) $\frac{1}{3}\cos\left(\frac{x}{3}-4\right)+\frac{7}{7x+9}+e^{\frac{x}{5}+3}$

11.6 Simple applications

So far in this section we have been using x as the variable of integration. In the case of applications, it is often convenient to use a different variable. For instance in the equation of motion the independent variable is time and the variable of integration is t.

In this section we discuss how integration is used to find the position and velocity of an object, given its acceleration and similar types of problems. Mathematically, this means that, starting with the derivative of a function, we must find the original function. Many common word which indicate derivative such as rate, growth, decay, marginal, change, varies, increase, decrease etc.

Example 11.10

If $f'(x)=3x^2-4x+5$ and $f(1)=3$, then find $f(x)$.

Solution

Given that $\quad f'(x)=\frac{d}{dx}\big(f(x)\big)=3x^2-4x+5$

Integrating on both sides with respect to x, we get

$$\int f'(x)\,dx=\int(3x^2-4x+5)\,dx$$
$$f(x)=x^3-2x^2+5x+c$$

To determine the constant of integration c, we have to apply the given information $f(1)=3$

$$f(1)=3\Rightarrow\ 3=(1)^3-2(1)^2+5(1)+c\Rightarrow\ c=-1$$

Thus $\quad f(x)=x^3-2x^2+5x-1.$

Example 11.11

A train started from Madurai Junction towards Coimbatore at 3pm (time $t=0$) with velocity $v(t)=20t+50$ kilometre per hour, where t is measured in hours. Find the distance covered by the train at 5pm.

Solution

In calculus terminology, velocity $v=\frac{ds}{dt}$ is rate of change of position with time, where s is the distance.The velocity of the train is given by

$$v(t)=20t+50$$

Therefore, $\frac{ds}{dt} = 20t + 50$

To find the distance function s one has to integrate the derivative function.

That is, $s = \int (20t + 50)\, dt$

$s = 10t^2 + 50t + c$

The distance covered by the train is zero when time is zero. Let us use this initial condition $s = 0$ at $t = 0$ to determine the value c of the constant of integration.

$\Rightarrow s = 10t^2 + 50t + c \quad \Rightarrow c = 0$

Therefore, $s = 10t^2 + 50t$

The distance covered by the train in 2 hours (5pm-3pm) is given by substituting $t = 2$ in the above equation, we get

$s = 10(2)^2 + 50(2) = 140$ km.

Example 11.12

The rate of change of weight of person w in kg with respect to their height h in centimetres is given approximately by $\frac{dw}{dh} = 4.364 \times 10^{-5} h^2$. Find weight as a function of height. Also find the weight of a person whose height is 150 cm.

Solution

The rate of change of weight with respect to height is

$$\frac{dw}{dh} = 4.364 \times 10^{-5} h^2$$

$$w = \int 4.364 \times 10^{-5} h^2 \; dh$$

$$w = 4.364 \times 10^{-5} \left(\frac{h^3}{3} \right) + c$$

One can obviously understand that the weight of a person is zero when height is zero.

Let us find the value c of the constant of integration by substituting the initial condition $w = 0$, at $h = 0$, in the above equation

$$w = 4.364 \times 10^{-5} \left(\frac{h^3}{3} \right) + c \Rightarrow c = 0$$

The required relation between weight and height of a person is

$$w = 4.364 \times 10^{-5} \left(\frac{h^3}{3} \right)$$

When the height h =150cm,

$$w = 4.364 \times 10^{-5} \left(\frac{150^3}{3} \right)$$

When the height h = 150cm, the weight is $w = 49kg$ (approximately)
Therefore, the weight of the person whose height 150cm is 49 kg.

Example 11.13

A tree is growing so that, after t - years its height is increasing at a rate of $\frac{18}{\sqrt{t}}$ cm per year. Assume that when $t = 0$, the height is 5 cm.

(i) Find the height of the tree after 4 years.

(ii) After how many years will the height be 149 cm?

Solution

The rate of change of height h with respect to time t is the derivative of h with respect to t.

$$\text{Therefore, } \frac{dh}{dt} = \frac{18}{\sqrt{t}} = 18t^{-\frac{1}{2}}$$

So, to get a general expression for the height, integrating the above equation with respect to t.

$$h = \int 18t^{-\frac{1}{2}} dt = 18(2t^{\frac{1}{2}}) + c = 36\sqrt{t} + c$$

Given that when $t = 0$, the height h = 5 cm.

$$5 = 0 + c \Rightarrow c = 5$$

$$h = 36\sqrt{t} + 5.$$

(i) To find the height of the tree after 4 years.

When t = 4 years,

$$h = 36\sqrt{t} + 5 \Rightarrow h = 36\sqrt{4} + 5 = 77$$

The height of the tree after 4 years is 77 cm

(ii) When h = 149cm

$$h = 36\sqrt{t} + 5 \Rightarrow 149 = 36\sqrt{t} + 5$$

$$\sqrt{t} = \frac{149 - 5}{36} = 4 \Rightarrow t = 16$$

Thus after 16 years the height of the tree will be 149 cm.

Example 11.14

At a particular moment, a student needs to stop his speedy bike to avoid a collision with the barrier ahead at a distance 40 metres away from him. Immediately he slows (retardation) the bike under braking at a rate of 8 metre/second^2. If the bike is moving at a speed of 24m/s, when the brakes are applied, would it stop before collision?

Solution

Let a be the acceleration, v be the velocity of the car, and s be the distance.

Stated in calculus terminology, velocity, $v = \frac{ds}{dt}$, is the rate of change of position with time, and acceleration, $a = \frac{dv}{dt}$, is rate of change of velocity with time.

The acceleration to be negative because if you take the direction of movement to be positive, then for a bike that is slowing down, its acceleration vector will be oriented in the opposite direction of its motion (retardation).

Given that the retardation of the car is 8 meter/second^2.

$$\text{Therefore, } a = \frac{dv}{dt} = -8 \text{ meter/second}^2.$$

$$\text{Therefore, } v = \int a\,dt = \int -8dt = -8t + c_1$$

$$v = -8t + c_1.$$

When the brakes are applied,

$$t = 0, \text{ and } v = 24\text{m/s}.$$

$$\text{So, } 24 = -8(0) + c_1 \Rightarrow c_1 = 24$$

$$\text{Therefore, } v = -8t + 24.$$

$$\text{That is, } \frac{ds}{dt} = -8t + 24.$$

It is required to find the distance, not the velocity, so need more integration in order.

$$s = \int v\,dt = \int (-8t + 24)dt$$

$$s = -4t^2 + 24t + c_2$$

To determine c_2, the stopping distance s is measured from where, and when, the brakes are applied so that at $t = 0,\ s = 0$.

$$s = -4t^2 + 24t + c_2 \Rightarrow 0 = -4(0)^2 + 24(0) + c_2 \Rightarrow c_2 = 0$$

$$s = -4t^2 + 24t$$

The stopping distance s could be evaluated if we knew the braking time. The time can be determined from the speed statement.

The bike stops when $v = 0$, $\Rightarrow v = -8t + 24 \quad \Rightarrow 0 = -8t + 24 \Rightarrow t = 3$.

When $t = 3$, we get

$$s = -4t^2 + 24t \Rightarrow s = -4(3)^2 + 24(3)$$

$$s = 36 \text{ metres} < 40 \text{ metres}$$

The bike stops at a distance 4 metres to the barrier.

EXERCISE 11.4

(1) If $f'(x) = 4x - 5$ and $f(2) = 1$, find $f(x)$.

(2) If $f'(x) = 9x^2 - 6x$ and $f(0) = -3$, find $f(x)$.

(3) If $f''(x) = 12x - 6$ and $f(1) = 30$, $f'(1) = 5$ find $f(x)$.

(4) A ball is thrown vertically upward from the ground with an initial velocity of 39.2 m/sec. If the only force considered is that attributed to the acceleration due to gravity, find

(i) how long will it take for the ball to strike the ground?

(ii) the speed with which will it strike the ground? and

(iii) how high the ball will rise?

(5) A wound is healing in such a way that t days since Sunday the area of the wound has been decreasing at a rate of $-\frac{6}{(t+2)^2}$ cm^2 per day where $0<t\leq 8$. If on Monday the area of the wound was 1.4 cm^2

(i) What was the area of the wound on Sunday?

(ii) What is the anticipated area of the wound on Thursday if it continues to heal at the same rate?

11.7 Methods of Integration

Integration is not as easy as differentiation. This is first due to its nature. Finding a derivative of a given function is facilitated by the fact that the differentiation itself has a constructive character. A derivative is simply defined as

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Suppose we are asked to find the derivative of $\log x$, we know in all details how to proceed in order to obtain the result.

When we are asked to find the integral of $\log x$, we have no constructive method to find integral or even how to start.

In the case of differentiation we use the laws of differentiation of several functions in order to find derivatives of their various combinations, like their sum, product, quotient, composition of functions etc.

There are very few such rules available in the theory of integration and their application is rather restricted. But the significance of these methods of integration is very great.

In every case one must learn to select the most appropriate method and use it in the most convenient form. This skill can only be acquired after long practice.

Already we have seen two important properties of integration. The following are the four important methods of integrations.

(1) Integration by decomposition into sum or difference.

(2) Integration by substitution.

(3) Integration by parts

(4) Integration by successive reduction.

BDV3YH

Here we discuss only the first three methods of integration and the other will be studied in higher classes.

11.7.1 Decomposition method

Sometimes it is very difficult to integrate the given function directly. But it can be integrated after decomposing it into a sum or difference of number of functions whose integrals are already known.

For example $\left(1-x^3\right)^2$, $\frac{x^2-x+1}{x^3}$, $\cos 5x \sin 3x$, $\cos^3 x$, $\frac{e^{2x}-1}{e^x}$, do not have direct formulae to integrate. But these functions can be decomposed into a sum or difference of functions, whose individual integrals are known. In most of the cases the given integrand will be any one of the algebraic, trigonometric or exponential forms, and sometimes combinations of these functions.

Example 11.15

Integrate the following with respect to x:

(i) $\left(1-x^3\right)^2$ (ii) $\frac{x^2-x+1}{x^3}$

Solution

(i)
$$\int\left(1-x^3\right)^2 dx = \int(1-2x^3+x^6)dx$$
$$= \int dx - 2\int x^3 dx + \int x^6 dx$$
$$= x - \frac{x^4}{2} + \frac{x^7}{7} + c.$$

(ii)
$$\int \frac{x^2-x+1}{x^3} dx = \int\left(\frac{x^2}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}\right) dx$$
$$= \int \frac{1}{x} dx - \int \frac{1}{x^2} dx + \int \frac{1}{x^3} dx.$$
$$\int \frac{x^2-x+1}{x^3} dx = \log|x| + \frac{1}{x} - \frac{1}{2x^2} + c.$$

Example 11.16

Integrate the following with respect to x:

(i) $\cos 5x \sin 3x$ (ii) $\cos^3 x$.

Solution

(i)
$$\int \cos 5x \sin 3x \, dx = \frac{1}{2}\int 2\cos 5x \sin 3x \, dx$$
$$= \frac{1}{2}\int\left(\sin 8x - \sin 2x\right) dx$$
$$\int \cos 5x \sin 3x \, dx = \frac{1}{2}\left(-\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right) + c.$$

(ii)
$$\int \cos^3 x \, dx = \frac{1}{4}\int\left(3\cos x + \cos 3x\right) dx$$
$$= \frac{1}{4}\left(3\sin x + \frac{\sin 3x}{3}\right) + c$$

Example 11.17

Integrate the following with respect to x:

(i) $\dfrac{e^{2x}-1}{e^x}$ (ii) $e^{3x}\left(e^{2x}-1\right)$.

Solution

(i)
$$\int \frac{e^{2x}-1}{e^x}dx = \int\left(\frac{e^{2x}}{e^x}-\frac{1}{e^x}\right)dx$$
$$= \int\left(e^x-e^{-x}\right)dx = e^x+e^{-x}+c.$$

(ii)
$$\int e^{3x}\left(e^{2x}-1\right)dx = \int\left(e^{5x}-e^{3x}\right)dx = \frac{e^{5x}}{5}-\frac{e^{3x}}{3}+c.$$

Example 11.18

Evaluate : $\displaystyle\int \frac{1}{\sin^2 x\cos^2 x}dx.$

Solution

$$\int \frac{1}{\sin^2 x\cos^2 x}dx = \int \frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}dx$$
$$= \int \frac{1}{\cos^2 x}dx+\int\frac{1}{\sin^2 x}dx$$
$$= \int \sec^2 x dx+\int \text{cosec}^2 x dx$$
$$= \tan x-\cot x+c.$$

Example 11.19

Evaluate : $\displaystyle\int \frac{\sin x}{1+\sin x}dx.$

Solution

$$\int \frac{\sin x}{1+\sin x}dx = \int\left(\frac{\sin x}{1+\sin x}\right)\left(\frac{1-\sin x}{1-\sin x}\right)dx$$
$$= \int \frac{\sin x-\sin^2 x}{1-\sin^2 x}dx = \int\frac{\sin x-\sin^2 x}{\cos^2 x}dx = \int\frac{\sin x}{\cos^2 x}dx-\int\frac{\sin^2 x}{\cos^2 x}dx$$
$$= \int \tan x\sec x dx-\int\tan^2 x dx$$
$$= \int \tan x\sec x dx-\int(\sec^2 x-1)dx$$
$$= \sec x-\tan x+x+c.$$

Example 11.20

Evaluate : $\displaystyle\int\sqrt{1+\cos 2x}\,dx.$

Solution

$$\int\sqrt{1+\cos 2x}\,dx = \int\sqrt{2\cos^2 x}\,dx = \sqrt{2}\int\cos x dx = \sqrt{2}\sin x+c$$

Example 11.21

Evaluate : $\int \frac{(x-1)^2}{x^3+x}dx.$

Solution

$$\begin{aligned}\int \frac{(x-1)^2}{x^3+x}dx &= \int \frac{x^2+1-2x}{x(x^2+1)}dx \\ &= \int\left(\frac{(x^2+1)}{x(x^2+1)} - \frac{2x}{x(x^2+1)}\right)dx \\ &= \int \frac{1}{x}dx - 2\int \frac{1}{1+x^2}dx \\ &= \log|x| - 2\tan^{-1}x + c.\end{aligned}$$

Example 11.22

Evaluate : $\int (\tan x + \cot x)^2 dx.$

Solution

$$\begin{aligned}\int (\tan x + \cot x)^2 dx &= \int [\tan^2 x + 2\tan x \cot x + \cot^2 x]\,dx \\ &= \int [(\sec^2 x - 1) + 2 + (\mathrm{cosec}^2 x - 1)]\,dx \\ &= \int (\sec^2 x + \mathrm{cosec}^2 x)\,dx \\ &= \tan x + (-\cot x) + c \\ &= \tan x - \cot x + c.\end{aligned}$$

Example 11.23

Evaluate : $\int \frac{1-\cos x}{1+\cos x}dx.$

Solution

$$\begin{aligned}\int \frac{1-\cos x}{1+\cos x}dx &= \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}dx = \int \tan^2\frac{x}{2}dx \\ &= \int\left(\sec^2\frac{x}{2} - 1\right)dx = \frac{\tan\frac{x}{2}}{\frac{1}{2}} - x + c \\ &= 2\tan\frac{x}{2} - x + c.\end{aligned}$$

Example 11.24

Evaluate : $\int \sqrt{1+\sin 2x}\,dx.$

Solution

$$\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{(\cos^2 x + \sin^2 x) + (2\sin x\cos x)}\,dx$$

$$= \int \sqrt{(\cos x+\sin x)^2}\,dx = \int (\cos x+\sin x)dx$$
$$= \sin x - \cos x + c$$

Example 11.25

Evaluate : $\int \frac{x^3+2}{x-1}dx.$

Solution

$$\int \frac{x^3+2}{x-1}dx = \int \frac{x^3-1+3}{x-1}dx = \int\left(\frac{x^3-1}{x-1}+\frac{3}{x-1}\right)dx$$
$$= \int\left[\frac{(x-1)(x^2+x+1)}{x-1}+\frac{3}{x-1}\right]dx$$
$$= \int\left(x^2+x+1+\frac{3}{x-1}\right)dx$$
$$= \frac{x^3}{3}+\frac{x^2}{2}+x+3\log|(x-1)|+c.$$

Example 11.26

Evaluate : (i) $\int a^x e^x dx$ (ii) $\int e^{x\log 2} e^x\, dx.$

Solution

(i) $$\int a^x e^x dx = \int (ae)^x dx = \frac{(ae)^x}{\log(ae)}+c$$

(ii) $$\int e^{x\log 2}e^x dx = \int e^{\log 2^x}e^x dx = \int 2^x e^x dx$$
$$= \int (2e)^x dx = \frac{(2e)^x}{\log(2e)}+c.$$

Example 11.27

Evaluate : $\int (x-3)\sqrt{x+2}\; dx.$

Solution

$$\int (x-3)\sqrt{x+2}\;dx = \int (x+2-5)\sqrt{x+2}\;dx$$
$$= \int (x+2)\sqrt{x+2}\;dx - 5\int\sqrt{x+2}\;dx$$
$$= \int (x+2)^{\frac{3}{2}}dx - 5\int (x+2)^{\frac{1}{2}}dx$$
$$= \frac{(x+2)^{\frac{5}{2}}}{\frac{5}{2}} - 5\frac{(x+2)^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= \frac{2}{5}(x+2)^{\frac{5}{2}} - \frac{10}{3}(x+2)^{\frac{3}{2}} + c.$$

Example 11.28

Evaluate : $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx.$

Solution

$$\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx = \int \frac{1}{\sqrt{x+1}+\sqrt{x}}\left[\frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right]dx$$

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{\left(\sqrt{x+1}^{2}\right)-\left(\sqrt{x}\right)^{2}}\,dx$$

$$= \int \frac{\sqrt{x+1}-\sqrt{x}}{x+1-x}\,dx = \int\left(\sqrt{x+1}-\sqrt{x}\right)dx$$

$$= \int \sqrt{x+1}dx - \int \sqrt{x}dx = \int (x+1)^{\frac{1}{2}}dx - \int x^{\frac{1}{2}}dx$$

$$= \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left[(x+1)^{\frac{3}{2}} - x^{\frac{3}{2}}\right] + c.$$

11.7.2 Decomposition by Partial Fractions

One of the important methods to evaluate integration is partial fractions. If the integrand is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, then the fraction need to be expressed in **partial fractions** before integration takes place. We will assume that we have a rational function $\frac{p(x)}{q(x)}$, $\left(q(x)\neq 0\right)$ in which degree of $p\,(x)$ < degree of $q\,(x)$. If this is not the case, we can always perform long division.

Example 11.29

Evaluate : (i) $\int \frac{3x+7}{x^2-3x+2}\,dx$ (ii) $\int \frac{x+3}{(x+2)^2(x+1)}\,dx.$

Solution

(i) $$\int \frac{3x+7}{x^2-3x+2}\,dx = \int \frac{13}{x-2}\,dx - \int \frac{10}{x-1}\,dx \quad \left\{\text{Resolving into partial fractions}\right.$$

$$= 13\log|x-2| - 10\log|x-1| + c$$

(ii) $$\int \frac{x+3}{(x+2)^2(x+1)}\,dx = \int \frac{-2}{x+2}\,dx - \int \frac{1}{(x+2)^2}\,dx + \int \frac{2}{x+1}\,dx \quad \left\{\text{Resolving into partial fractions}\right.$$

$$= -2\int \frac{1}{x+2}\,dx - \int \frac{1}{(x+2)^2}\,dx + 2\int \frac{1}{x+1}\,dx$$

$$= -2\log|x+2| - \int (x+2)^{-2}\,dx + 2\log|x+1| + c$$

$$= -2\log|x+2| + \frac{1}{x+2} + 2\log|x+1| + c.$$

EXERCISE 11.5

Integrate the following functions with respect to x :

(1) $\dfrac{x^3+4x^2-3x+2}{x^2}$ (2) $\left(\sqrt{x}+\dfrac{1}{\sqrt{x}}\right)^2$ (3) $(2x-5)(36+4x)$

(4) $\cot^2 x+\tan^2 x$ (5) $\dfrac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$ (6) $\dfrac{\cos 2x}{\sin^2 x\cos^2 x}$

(7) $\dfrac{3+4\cos x}{\sin^2 x}$ (8) $\dfrac{\sin^2 x}{1+\cos x}$ (9) $\dfrac{\sin 4x}{\sin x}$

(10) $\cos 3x\cos 2x$ (11) $\sin^2 5x$ (12) $\dfrac{1+\cos 4x}{\cot x-\tan x}$

(13) $e^{x\log a}e^x$ (14) $(3x+4)\sqrt{3x+7}$ (15) $\dfrac{8^{1+x}+4^{1-x}}{2^x}$

(16) $\dfrac{1}{\sqrt{x+3}-\sqrt{x-4}}$ (17) $\dfrac{x+1}{(x+2)(x+3)}$ (18) $\dfrac{1}{(x-1)(x+2)^2}$

(19) $\dfrac{3x-9}{(x-1)(x+2)(x^2+1)}$ (20) $\dfrac{x^3}{(x-1)(x-2)}$

11.7.3 Method of substitution or change of variable

The method of substitution in integration is similar to finding the derivative of function of function in differentiation. By using a suitable substitution, the variable of integration is changed to new variable of integration which will be integrated in an easy manner.

We know that, if u is a function of x then $\dfrac{du}{dx}=u'$.

Hence we can write $\int f(u)u'dx=\int f(u)du$

Thus, $\int f[g(x)]g'(x)dx=\int f(u)du,$ where $u=g(x)$

The success of the above method depends on the selection of suitable substitution either $x=\phi(u)$ or $u=g(x)$.

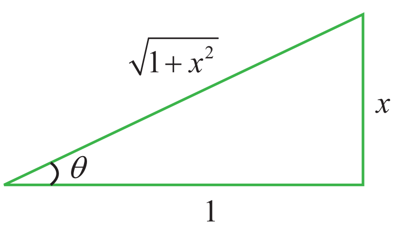
Note 11.2

The substitution for the variable of integration is in trigonometric function, use a rough diagram to find the re -substitution value for it. Suppose the variable of integration x is substituted as $x=\tan\theta$. After integration suppose the solution is $\sec\theta+\operatorname{cosec}\theta$

For example, if $x=\tan\theta,$ then from the figure

$\operatorname{cosec}\theta=\left(\dfrac{\sqrt{1+x^2}}{x}\right),$

$\sec\theta=\left(\dfrac{\sqrt{1+x^2}}{1}\right)$



Then $\sec\theta+\operatorname{cosec}\theta=\sqrt{1+x^2}+\left(\dfrac{\sqrt{1+x^2}}{x}\right).$

Example 11.30

Evaluate the following integrals :

(i) $\int 2x\sqrt{1+x^2}\,dx$ (ii) $\int e^{-x^2} x\,dx$ (iii) $\int \frac{\sin x}{1+\cos x}\,dx$

(iv) $\int \frac{1}{1+x^2}\,dx$ (v) $\int x(a-x)^8\,dx$

Solution

(i) $\int 2x\sqrt{1+x^2}\,dx$

Putting $1+x^2 = u$, then $2x\,dx = du$

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du$$

$$= \int u^{\frac{1}{2}} du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c\,.$$

(ii) $\int e^{-x^2} x\,dx$

Putting $x^2 = u$ then $2x\,dx = du$

Therefore, $\int e^{-x^2} x\,dx = \int e^{-u}\frac{du}{2}$

$$= \frac{1}{2}\int e^{-u} du = \frac{1}{2}(-e^{-u}) + c = -\frac{1}{2}e^{-u} + c = -\frac{1}{2}e^{-x^2} + c.$$

(iii) $\int \frac{\sin x}{1+\cos x}\,dx$

Putting $1+\cos x = u$, then $-\sin x\,dx = du$

Therefore, $\int \frac{\sin x}{1+\cos x}\,dx = \int \frac{-du}{u} = -\log|u| + c = -\log|1+\cos x| + c.$

(iv) $\int \frac{1}{1+x^2}\,dx$

Putting $x = \tan u$, then $dx = \sec^2 u\,du$

$$\int \frac{1}{1+x^2}\,dx = \int \frac{\sec^2 u}{1+\tan^2 u}\,du = \int \frac{\sec^2 u}{\sec^2 u}\,du = \int du = u + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + c.$$

(v) $\int x(a-x)^8\,dx$

Putting $u = a-x$, then $du = -dx$

$$\int x(a-x)^8\,dx = \int x(a-x)^8\,dx$$

$$= \int (a-u)(u)^8(-du)$$

$$= \int \left(-a(u)^8 + u^9\right) du$$

$$= \int u^9 du - a\int u^8 du$$

$$= \frac{u^{10}}{10} - a\frac{u^9}{9} + c$$

$$\int x(a-x)^8\, dx = \frac{(a-x)^{10}}{10} - \frac{a(a-x)^9}{9} + c.$$

11.7.4 Important Results

(1) $$\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$$

(2) $$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c, \quad n \neq -1$$

Proof

(1) Let $I = \int \frac{f'(x)}{f(x)}dx$

Putting $f(x) = u$ then $f'(x)dx = du$

Thus, $I = \int \frac{du}{u} = \log|u| + c$

Therefore, $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c.$

(2) Let $I = \int f'(x)[f(x)]^n dx$

Putting $f(x) = u$ then $f'(x)dx = du$

Thus, $I = \int u^n du = \frac{u^{n+1}}{n+1} + c$

Therefore, $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c.$

Example 11.31

Integrate the following with respect to x.

(i) $\int \tan x\, dx$ (ii) $\int \cot x\, dx$ (iii) $\int \operatorname{cosec} x\, dx$ (iv) $\int \sec x\, dx$

Solution

(i) Let $I = \int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$

Putting $\cos x = u$ then, $-\sin x\, dx = du$

Thus, $I = \int -\frac{1}{u}du = -\log|u| + c = -\log|\cos x| + c = \log|\sec x| + c.$

(ii) Let $I = \int \cot x\, dx = \int \frac{\cos x}{\sin x}dx$

Putting $\sin x = u$ then, $\cos x\, dx = du$

Thus, $I = \int \frac{1}{u}du = \log|u| + c = \log|\sin x| + c.$

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(iii) Let $I = \int \operatorname{cosec} x dx = \dfrac{\int \operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

$$= \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{\operatorname{cosec} x - \cot x} dx$$

Putting $\operatorname{cosec} x - \cot x = u$, then $(\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x) dx = du$

$$\text{Thus, } I = \int \frac{1}{u} du = \log|u| + c = \log|\operatorname{cosec} x - \cot x| + c.$$

(iv) Let $I = \int \sec x dx = \int \dfrac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \dfrac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$

Putting $\sec x + \tan x = u$, then $(\sec^2 x + \sec x \tan x) dx = du$

$$\text{Thus, } I = \int \frac{1}{u} du = \log|u| + c = \log|\sec x + \tan x| + c$$

Therefore, $\int \sec x dx = \log|\sec x + \tan x| + c.$

Thus the following are the important standard results.

(1)	$\int \tan x \, dx = \log\|\sec x\| + c$
(2)	$\int \cot x \, dx = \log\|\sin x\| + c$
(3)	$\int \operatorname{cosec} x \, dx = \log\|\operatorname{cosec} x - \cot x\| + c$
(4)	$\int \sec x \, dx = \log\|\sec x + \tan x\| + c$

Example 11.32

Integrate the following with respect to x.

(i) $\displaystyle\int \frac{2x+4}{x^2+4x+6} dx$ (ii) $\displaystyle\int \frac{e^x}{e^x - 1} dx$ (iii) $\displaystyle\int \frac{1}{x \log x} dx$

(iv) $\displaystyle\int \frac{\sin x + \cos x}{\sin x - \cos x} dx$ (v) $\displaystyle\int \frac{\cos 2x}{(\sin x + \cos x)^2} dx$

Solution

(i) Let $I = \displaystyle\int \frac{2x+4}{x^2+4x+6} dx$

Putting $x^2 + 4x + 6 = u$, then $(2x+4) dx = du$

$$\text{Thus, } I = \int \frac{du}{u} = \log|u| + c = \log|x^2 + 4x + 6| + c$$

$$\text{Therefore, } \int \frac{2x+4}{x^2+4x+6} dx = \log|x^2 + 4x + 6| + c.$$

(ii) $$\text{Let } I = \int \frac{e^x}{e^x - 1}\,dx.$$

$$\text{Putting } e^x - 1 = u, \text{ then } e^x dx = du$$

$$\text{Thus, } I = \int \frac{du}{u} = \log|u| + c = \log|e^x - 1| + c$$

$$\text{Therefore, } \int \frac{e^x}{e^x - 1}\,dx = \log|e^x - 1| + c.$$

(iii) $$\text{Let } I = \int \frac{1}{x\log x}\,dx.$$

$$\text{Putting } \log x = u, \text{ then } \frac{1}{x}dx = du$$

$$\text{Thus, } I = \int \frac{du}{u} = \log|u| + c = \log|\log x| + c$$

$$\text{Therefore, } \int \frac{1}{x\log x}\,dx = \log|\log x| + c.$$

(iv) $$\text{Let } I = \int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx.$$

$$\text{Putting } \sin x - \cos x = u, \text{ then } (\cos x + \sin x)\,dx = du$$

$$\text{Thus, } I = \int \frac{du}{u} = \log|u| + c = \log|\sin x - \cos x| + c$$

$$\text{Therefore, } \int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx = \log|\sin x - \cos x| + c$$

(v) $$\text{Let } I = \int \frac{\cos 2x}{(\sin x + \cos x)^2}\,dx = \int \frac{\cos 2x}{1 + \sin 2x}\,dx.$$

$$\text{Putting } 1 + \sin 2x = u, \text{ then } 2\cos 2x\,dx = du$$

$$\text{Thus, } I = \int \frac{du}{2u} = \frac{1}{2}\log|u| + c = \frac{1}{2}\log|1 + \sin 2x| + c.$$

EXERCISE 11.6

Integrate the following with respect to x

(1) $\dfrac{x}{\sqrt{1+x^2}}$ (2) $\dfrac{x^2}{1+x^6}$ (3) $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ (4) $\dfrac{10x^9 + 10^x \log_e 10}{10^x + x^{10}}$

(5) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$ (6) $\dfrac{\cot x}{\log(\sin x)}$ (7) $\dfrac{\operatorname{cosec} x}{\log\left(\tan\frac{x}{2}\right)}$ (8) $\dfrac{\sin 2x}{a^2 + b^2\sin^2 x}$

(9) $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$ (10) $\dfrac{\sqrt{x}}{1+\sqrt{x}}$ (11) $\dfrac{1}{x\log x\log(\log x)}$ (12) $\alpha\beta x^{\alpha-1}e^{-\beta x^{\alpha}}$

(13) $\tan x\sqrt{\sec x}$ (14) $x(1-x)^{17}$ (15) $\sin^5 x\cos^3 x$ (16) $\dfrac{\cos x}{\cos(x-a)}$

11.7.5 Integration by parts

Integration by parts method is generally used to find the integral when the integrand is a product of two different types of functions or a single logarithmic function or a single inverse trigonometric function or a function which is not integrable directly. From the formula for derivative of product of two functions we obtain this useful method of integration.

If u and v are two differentiable functions then we have

$$d(uv) = vdu + udv$$
$$udv = d(uv) - vdu$$

Integrating

$$\int udv = \int d(uv) - \int vdu$$
$$\int udv = uv - \int vdu$$

$\int udv$ in terms of another integral $\int vdu$ and does not give a final expression for the integral $\int udv$. It only partially solves the problem of integrating the product $u\,dv$. Hence the term '**Partial Integration**' has been used in many European countries. The term "**Integration by Parts**" is used in many other countries as well as in our own.

The success of this method depends on the proper choice of u

(i) If integrand contains any non integrable functions directly from the formula, like $\log x$, $\tan^{-1} x$ etc., we have to take these non integrable functions as u and other as dv.

(ii) If the integrand contains both the integrable function, and one of these is x^n (where n is a positive integer) then take $u = x^n$.

(iii) For other cases the choice of u is ours.

Example 11.33

Evaluate the following integrals

(i) $\int xe^x dx$ (ii) $\int x\cos x\, dx$ (iii) $\int \log x\, dx$ (iv) $\int \sin^{-1} x\, dx$

Solution

(i) Let $I = \int xe^x dx.$

Since x is an algebraic function and e^x is an exponential function,

so take $u = x$ then $du = dx$

$dv = e^x dx \Rightarrow v = e^x$

Applying Integration by parts, we get

$$\int udv = uv - \int vdu$$
$$\Rightarrow \int xe^x dx = xe^x - \int e^x dx$$

That is, $\int xe^x dx = xe^x - e^x + c.$

(ii) Let $I = \int x\cos x\, dx$

Since x is an algebraic function and $\cos x$ is a trigonometric function, so take $u = x$ then $du = dx$

$dv = \cos x\,dx \Rightarrow v = \sin x$

Applying Integration by parts, we get

$$\int udv = uv - \int vdu$$

$$\Rightarrow \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$\Rightarrow \int x\cos x\,dx = x\sin x + \cos x + c$$

(iii) Let $I = \int \log x\,dx$

Take $u = \log x$ then $du = \frac{1}{x}dx$

$dv = dx \Rightarrow v = x$

Applying Integration by parts, we get

$$\int udv = uv - \int vdu$$

$$\Rightarrow \int \log x\,dx = x\log x - \int x\frac{1}{x}dx$$

$$\Rightarrow \int \log x\,dx = x\log x - x + c$$

(iv) Let $I = \int \sin^{-1} x\,dx$

$u = \sin^{-1}(x), dv = dx$

Then $du = \frac{1}{\sqrt{1-x^2}}, v = x$

$$\int \sin^{-1} xdx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$\int \sin^{-1} xdx = x\sin^{-1} x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}, \text{where } t = 1-x^2$$

$$= x\sin^{-1} x + \sqrt{t} + c$$

$$= x\sin^{-1} x + \sqrt{1-x^2} + c$$

Example 11.34

Evaluate : $\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Solution

Let $I = \int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$

Therefore, $I = \int \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)\sec^2\theta\,d\theta$

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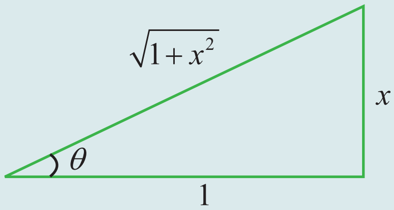
$$= \int \tan^{-1}(\tan 2\theta)\sec^2\theta \, d\theta$$
$$= \int 2\theta \sec^2\theta \, d\theta$$
$$= 2\int (\theta)(\sec^2\theta \, d\theta)$$

Applying integration by parts



$$I = 2\left[\theta\tan\theta - \int \tan\theta \, d\theta\right]$$
$$= 2(\theta\tan\theta - \log|\sec\theta|) + c$$
$$\int \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx = 2x\tan^{-1}x - 2\log|\sqrt{1+x^2}| + c$$

$\tan\theta = x$

$\sec\theta = \sqrt{1+x^2}$

11.7.6 Bernoulli's formula for Integration by Parts

If u and v are functions of x, then the Bernoulli's rule is

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

where $u', u'', u''', \ldots$ are successive derivatives of u and

$v, v_1, v_2, v_3, \cdots$ are successive integrals of dv

Bernoulli's formula is advantageously applied when $u = x^n$ (n is a positive integer)

For the following problems we have to apply the integration by parts two or more times to find the solution. In this case Bernoulli's formula helps to find the solution easily.

Example 11.35

Integrate the following with respect to x.

(i) x^2e^{5x} (ii) $x^3\cos x$ (iii) x^3e^{-x}

Solution

(i) $\int x^2e^{5x}dx.$

	$dv = e^{5x}dx$
$u = x^2$	$v = \frac{e^{5x}}{5}$
$u' = 2x$	$v_1 = \frac{e^{5x}}{5^2}$
$u'' = 2$	$v_2 = \frac{e^{5x}}{5^3}$
$u''' = 0$	$v_3 = \frac{e^{5x}}{5^4}$

Applying Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int x^2e^{5x}\,dx = \left(x^2\right)\left(\frac{e^{5x}}{5}\right) - (2x)\left(\frac{e^{5x}}{5^2}\right) + (2)\left(\frac{e^{5x}}{5^3}\right) - (0)\left(\frac{e^{5x}}{5^4}\right) + 0 + \cdots + 0 + c$$

$$= \frac{x^2e^{5x}}{5} - \frac{2xe^{5x}}{25} + \frac{2e^{5x}}{125} + c.$$

(ii) $\int x^3\cos x\,dx.$

	$dv = \cos x\,dx$
$u = x^3,$	$v = \sin x$
$u' = 3x,^2$	$v_1 = -\cos x$
$u'' = 6x,$	$v_2 = -\sin x$
$u''' = 6,$	$v_3 = \cos x$

Applying Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int x^3\cos x\,dx = \left(x^3\right)(\sin x) - \left(3x^2\right)(-\cos x) + (6x)(-\sin x) - (6)(\cos x) + c$$

$$= x^3\sin x + 3x^2\cos x - 6x\sin x - 6\cos x + c.$$

(iii) $\int x^3 e^{-x} dx.$

Applying Bernoulli's formula

$$\int u dv = uv - u'v_1 + u''v_2 - \cdots$$

$$\int x^3 e^{-x}\, dx = \left(x^3\right)\left(-e^{-x}\right) - \left(3x^2\right)\left(e^{-x}\right) + \left(6x\right)\left(-e^{-x}\right) - \left(6\right)\left(e^{-x}\right) + c$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c.$$

	$dv = e^{-x} dx$
$u = x^3,$	$v = -e^{-x}$
$u' = 3x^2,$	$v_1 = +e^{-x}$
$u'' = 6x,$	$v_2 = -e^{-x}$
$u''' = 6,$	$v_3 = e^{-x}$

EXERCISE 11.7

Integrate the following with respect to x:

(1) (i) $9xe^{3x}$ (ii) $x\sin 3x$ (iii) $25xe^{-5x}$ (iv) $x\sec x\tan x$

(2) (i) $x\log x$ (ii) $27\ x^2e^{3x}$ (iii) $x^2\cos x$ (iv) $x^3\sin x$

(3) (i) $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}}$ (ii) $x^5e^{x^2}$ (iii) $\tan^{-1}\left(\dfrac{8x}{1-16x^2}\right)$ (iv) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

11.7.8 Integrals of the form (i) $\int e^{ax}\sin bx\,dx$ (ii) $\int e^{ax}\cos bx\,dx$

The following examples illustrate that there are some integrals whose integration continues forever. Whenever we integrate function of the form $e^{ax}\cos bx$ or $e^{ax}\sin bx$, we have to apply the Integration by Parts rule twice to get the similar integral on both sides to solve.

Result 11.1

(i) $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx] + c$$

(ii) $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx] + c$$

Proof : (i)

$$\text{Let } I = \int e^{ax}\sin bx\,dx$$

$$\text{Take } u = \sin bx;\, du = b\cos bx\,dx$$

$$dv = e^{ax};\, v = \frac{e^{ax}}{a},$$

Applying Integration by parts, we get

$$I = \frac{e^{ax}}{a}\sin bx - \int \frac{e^{ax}}{a} b\cos bx\,dx$$

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\,dx$$

$$\text{Take } u = \cos bx;\, du = -b\sin bx\,dx,$$

$$dv = e^{ax};\, v = \frac{e^{ax}}{a},$$

Again applying integration by parts, we get

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx + \int \frac{e^{ax}}{a} b\sin bx\,dx\right]$$

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$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\frac{e^{ax}}{a}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx$$

$$I = \frac{e^{ax}}{a}\sin bx - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}I$$

$$\left(1+\frac{b^2}{a^2}\right)I = \frac{ae^{ax}\sin bx - be^{ax}\cos bx}{a^2}$$

$$\left(\frac{a^2+b^2}{a^2}\right)I = \frac{e^{ax}[a\sin bx - b\cos bx]}{a^2}$$

Therefore, $I = \dfrac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$

Therefore, $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$

Similarly, $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c}$$

$$\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c}$$

Caution

In applying integration by parts to specific integrals, the pair of choice for u and dv once initially assumed should be maintained for the successive integrals on the right hand side. (See the above two examples). The pair of choice should not be interchanged.

Examples 11.36

Evaluate the following integrals

(i) $\displaystyle\int e^{3x}\cos 2x\,dx$ (ii) $\displaystyle\int e^{-5x}\sin 3x\,dx$

(i) $\displaystyle\int e^{3x}\cos 2x\,dx$

Using the formula

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$$

For $a=3$ and $b=2$, we get

$$\int e^{3x}\cos 2x\,dx = \left(\frac{e^{3x}}{3^2+2^2}\right)(3\cos 2x + 2\sin 2x)+c$$

$$= \left(\frac{e^{3x}}{13}\right)(3\cos 2x + 2\sin 2x)+c.$$

(ii) $\int e^{-5x} \sin 3x\, dx$

Using the formula

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b\cos bx] + c$$

for $a = -5,\ b = 3$, we get

$$\int e^{-5x} \sin 3x\, dx = \left(\frac{e^{-5x}}{(-5)^2+3^2}\right)(-5\sin 3x - 3\cos 3x) + c$$

$$\int e^{-5x} \sin 3x\, dx = -\left(\frac{e^{-5x}}{34}\right)(5\sin 3x + 3\cos 3x) + c$$

EXERCISE 11.8

Integrate the following with respect to x

(1) (i) $e^{ax} \cos bx$ (ii) $e^{2x} \sin x$ (iii) $e^{-x} \cos 2x$

(2) (i) $e^{-3x} \sin 2x$ (ii) $e^{-4x} \sin 2x$ (iii) $e^{-3x} \cos x$

Result 11.2

$$\int e^x[f(x) + f'(x)]dx = e^x f(x) + c$$

Proof

$$\text{Let } I = \int e^x[f(x) + f'(x)]dx$$

$$= \int e^x f(x)dx + \int e^x f'(x)dx$$

Take $u = f(x);\ du = f'(x)\,dx$, in the first integral

$$dv = e^x;\ v = e^x,$$

$$\text{That is, } I = e^x f(x) - \int e^x f'(x)dx + \int e^x f'(x)dx + c$$

$$\text{Therefore, } I = e^x f(x) + c.$$

Examples 11.37

Evaluate the following integrals

(i) $\int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ (ii) $\int e^x(\sin x + \cos x)\, dx$ (iii) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution

(i)

$$\text{Let } I = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$\text{Take } f(x) = \frac{1}{x}, \text{ then } f'(x) = \frac{-1}{x^2}$$

This is of the form $\int e^x[f(x)+f'(x)]dx$

$$\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = e^x\frac{1}{x}+c.$$

(ii) Let $I = \int e^x(\sin x+\cos x)\,dx$

Take $f(x) = \sin x$, then $f'(x)=\cos x$

This is of the form $\int e^x[f(x)+f'(x)]\,dx$

$$\int e^x(\sin x+\cos x)\,dx = e^x\sin x+c.$$

(iii) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

$$\text{Let } I = \int e^x\frac{(1-x)^2}{\left(1+x^2\right)^2}dx$$

$$= \int e^x\frac{\left(1+x^2-2x\right)}{\left(1+x^2\right)^2}dx$$

$$= \int e^x\left(\frac{1}{\left(1+x^2\right)}-\frac{2x}{\left(1+x^2\right)^2}\right)dx$$

$$\text{If } f(x) = \frac{1}{\left(1+x^2\right)},\ \text{then}\ \ f'(x)=-\frac{2x}{\left(1+x^2\right)^2}$$

Using $\int e^x\left(f(x)+f'(x)\right)dx=e^x f(x)+c$

$$\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x\left(\frac{1}{\left(1+x^2\right)}-\frac{2x}{\left(1+x^2\right)^2}\right)dx = e^x\frac{1}{\left(1+x^2\right)}+c.$$

EXERCISE 11.9

Integrate the following with respect to x:

(1) $e^x(\tan x+\log\sec x)$ (2) $e^x\left(\frac{x-1}{2x^2}\right)$ (3) $e^x\sec x(1+\tan x)$

(4) $e^x\left(\frac{2+\sin 2x}{1+\cos 2x}\right)$ (5) $e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)$ (6) $\frac{\log x}{(1+\log x)^2}$

11.7.9 Integration of Rational Algebraic Functions

In this section we are going to discuss how to integrate the rational algebraic functions whose numerator and denominator contains some positive integral powers of x with constant coefficients.

Type I

Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2 - a^2}$, $\int \frac{dx}{\sqrt{a^2 \pm x^2}}$, $\int \frac{dx}{\sqrt{x^2 - a^2}}$

(i) $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$$

(ii) $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$$

(iii) $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$$

(iv) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$$

(v) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + c$$

(vi) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + c$$

Proof

(i) Let $I = \int \frac{dx}{a^2 - x^2}$.

$$= \int \frac{dx}{(a-x)(a+x)}$$

$$= \frac{1}{2a} \int \left[\frac{1}{a+x} + \frac{1}{a-x}\right] dx \qquad \left\{\text{Resolving into partial fractions}\right.$$

$$= \frac{1}{2a}\left[\log|a+x| - \log|a-x|\right] + c$$

$$= \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + c$$

(ii) Let $I = \int \frac{dx}{x^2 - a^2}$.

$$= \int \frac{dx}{(x-a)(x+a)}$$

$$= \frac{1}{2a} \int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx \qquad \left\{\text{Resolving into partial fractions}\right.$$

$$= \frac{1}{2a}[\log|x-a| - \log|x+a|] + c$$

$$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

Therefore $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

(iii) Let $I = \int \frac{dx}{a^2+x^2}$.

Putting $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\frac{x}{a}$

$dx = a\sec^2\theta$

$$I = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta}d\theta = \int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}d\theta = \int \frac{a\sec^2\theta}{a^2\sec^2\theta}d\theta = \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Therefore, $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(iv) Let $I = \int \frac{dx}{\sqrt{a^2-x^2}}$.

Putting $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a\cos\theta$

$$I = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}d\theta = \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}d\theta = \int \frac{a\cos\theta}{a\cos\theta}d\theta = \int d\theta$$

$$= \theta + c = \sin^{-1}\left(\frac{x}{a}\right) + c$$

Therefore, $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

(v) Let $I = \int \frac{dx}{\sqrt{x^2-a^2}}$

Putting $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$dx = a\sec\theta\tan\theta d\theta$

$$I = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2\sec^2\theta - a^2}}d\theta = \int \frac{a\sec\theta\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}}d\theta = \int \frac{a\sec\theta\tan\theta}{a\tan\theta}d\theta = \int \sec\theta d\theta$$

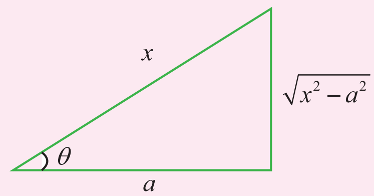
$$= \log|\sec\theta + \tan\theta| + c$$

$$= \log\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c$$

$$= \log\left|x+\sqrt{x^2-a^2}\right|-\log a+c$$

$$= \log\left|x+\sqrt{x^2-a^2}\right|+c_1 \text{ where } c_1=c-\log a$$

Therefore, $\displaystyle\int\frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|+c_1$



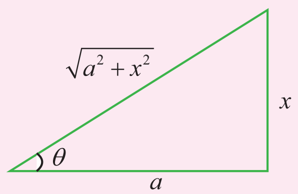
$\sec\theta=\dfrac{x}{a}$

$\tan\theta=\dfrac{\sqrt{x^2-a^2}}{a}$

(vi) Let $I=\displaystyle\int\frac{dx}{\sqrt{a^2+x^2}}$

Putting $x = a\tan\theta \Rightarrow \theta=\tan^{-1}\left(\dfrac{x}{a}\right)$

$dx = a\sec^2\theta d\theta$



$\tan\theta=\dfrac{x}{a}$

$\sec\theta=\dfrac{\sqrt{a^2+x^2}}{a}$

$$I = \int\frac{a\sec^2\theta}{\sqrt{a^2\tan^2\theta+a^2}}d\theta=\int\frac{a\sec^2\theta}{\sqrt{a^2(\tan^2\theta+1)}}d\theta$$

$$= \int\frac{a\sec^2\theta}{a\sec\theta}d\theta=\int\sec\theta d\theta$$

$$= \log|\sec\theta+\tan\theta|+c$$

$$= \log\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}+1}\right|+c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|-\log a+c$$

$$= \log\left|x+\sqrt{x^2+a^2}\right|+c_1 \text{ where } c_1=c-\log a$$

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \log\left|x+\sqrt{x^2+a^2}\right|+c_1$$

Remark: Remember the following useful substitution of the given integral as a functions of a^2-x^2, a^2+x^2 and x^2-a^2.

Given	Substitution
a^2-x^2	$x=a\sin\theta$
a^2+x^2	$x=a\tan\theta$
x^2-a^2	$x=a\sec\theta$

Examples 11.38

Evaluate the following integrals

(i) $\int \frac{1}{(x-2)^2+1}dx$ (ii) $\int \frac{x^2}{x^2+5}dx$ (iii) $\int \frac{1}{\sqrt{1+4x^2}}dx$ (iv) $\int \frac{1}{\sqrt{4x^2-25}}dx$

Solution

(i) Let $I = \int \frac{1}{(x-2)^2+1}dx = \int \frac{1}{(x-2)^2+1^2}dx$

Putting $x - 2 = t \Rightarrow dx = dt$

Thus, $I = \int \frac{1}{t^2+1^2}dt = \tan^{-1}(t)+c = \tan^{-1}(x-2)+c$

(ii) Let $I = \int \frac{x^2}{x^2+5}dx$

$$= \int \frac{x^2+5-5}{x^2+5}dx = \int\left(1-\frac{5}{x^2+5}\right)dx = \int dx - \int \frac{5}{x^2+5}dx$$

$$= x-5\int \frac{1}{x^2+\left(\sqrt{5}\right)^2}dx$$

$$= x-5\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$$

$$I = x-\sqrt{5}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$$

(iii) Let $I = \int \frac{1}{\sqrt{1+4x^2}}dx = \int \frac{1}{\sqrt{1+(2x)^2}}dx$

Putting $2x = t \Rightarrow 2\,dx = dt \Rightarrow dx = \frac{1}{2}dt$

Thus, $I = \frac{1}{2}\int \frac{1}{\sqrt{1^2+t^2}}dt$

$$I = \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right|+c = \frac{1}{2}\log\left|2x+\sqrt{(2x)^2+1}\right|+c$$

$$I = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right|+c$$

(iv) Let $I = \int \frac{1}{\sqrt{4x^2-25}}dx = \int \frac{1}{\sqrt{(2x)^2-25}}dx$

Putting $2x = t \Rightarrow 2\,dx = dt \Rightarrow dx = \frac{1}{2}dt$

$$\text{Thus, } I = \frac{1}{2}\int \frac{1}{\sqrt{t^2-5^2}}\,dt$$

$$= \frac{1}{2}\log\left|t+\sqrt{t^2-5^2}\right|+c$$

$$I = \frac{1}{2}\log\left|2x+\sqrt{4x^2-25}\right|+c$$

Type II

Integrals of the form $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

First we express ax^2+bx+c as sum or difference of two square terms that is, any one of the forms to Type I. The following rule is used to express the expression ax^2+bx+c as a sum or difference of two square terms.

(1) Make the coefficient of x^2 as unity.

(2) Completing the square by adding and subtracting the square of half of the coefficient of x.

$$\text{That is, } ax^2+bx+c = a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right]$$

Examples 11.39

Evaluate the following integrals

(i) $\int \frac{1}{x^2-2x+5}\,dx$ (ii) $\int \frac{1}{\sqrt{x^2+12x+11}}\,dx$ (iii) $\int \frac{1}{\sqrt{12+4x-x^2}}\,dx$

Solution

(i)

$$\int \frac{1}{x^2-2x+5}\,dx = \int \frac{1}{x^2-2(1)x+(1)^2+4}\,dx$$

$$= \int \frac{1}{(x-1)^2+2^2}\,dx$$

$$\int \frac{1}{x^2-2x+5}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$$

(ii)

$$\int \frac{1}{\sqrt{x^2+12x+11}}\,dx = \int \frac{1}{\sqrt{(x+6)^2-25}}\,dx$$

$$= \int \frac{1}{\sqrt{(x+6)^2-5^2}}\,dx$$

$$= \log\left|x+6+\sqrt{(x+6)^2-5^2}\right|+c$$

Therefore, $\int \frac{1}{\sqrt{x^2+12x+11}}\,dx = \log\left|x+6+\sqrt{x^2+12x+11}\right|+c$

(iii) $$\int \frac{1}{\sqrt{12+4x-x^2}}\,dx = \int \frac{1}{\sqrt{12-(x^2-4x)}}\,dx$$

$$= \int \frac{1}{\sqrt{12-\left\{(x-2)^2-4\right\}}}\,dx$$

$$= \int \frac{1}{\sqrt{4^2-(x-2)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{(x-2)}{4}\right)+c$$

EXERCISE 11.10

Find the integrals of the following :

(1) (i) $\frac{1}{4-x^2}$ (ii) $\frac{1}{25-4x^2}$ (iii) $\frac{1}{9x^2-4}$

(2) (i) $\frac{1}{6x-7-x^2}$ (ii) $\frac{1}{(x+1)^2-25}$ (iii) $\frac{1}{\sqrt{x^2+4x+2}}$

(3) (i) $\frac{1}{\sqrt{(2+x)^2-1}}$ (ii) $\frac{1}{\sqrt{x^2-4x+5}}$ (iii) $\frac{1}{\sqrt{9+8x-x^2}}$

Type III

Integrals of the form $\int \frac{px+q}{ax^2+bx+c}\,dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

To evaluate the above integrals, first we write

$$px+q = A\frac{d}{dx}(ax^2+bx+c)+B$$

$$px+q = A(2ax+b)+B$$

Calculate the values of A and B, by equating the coefficients of like powers of x on both sides

(i) The given first integral can be written as

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \int \frac{A(2ax+b)+B}{ax^2+bx+c}\,dx$$

$$= A\int \frac{2ax+b}{ax^2+bx+c}\,dx + B\int \frac{1}{ax^2+bx+c}\,dx$$

(The first integral is of the form $\int \frac{f'(x)}{f(x)}\,dx$)

$$= A\log|ax^2+bx+c| + B\int \frac{1}{ax^2+bx+c}\,dx$$

The second term on the right hand side can be evaluated using the previous types.

(ii) The given second integral can be written as

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = \int \frac{A(2ax+b)+B}{\sqrt{ax^2+bx+c}}\,dx$$

$$= A\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

(The first integral is of the form $\int f'(x)[f(x)]^n\,dx$)

$$= A\left(2\sqrt{ax^2+bx+c}\right) + B\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

The second term on the right hand side can be evaluated using the previous types.

Examples 11.40

Evaluate the following integrals

(i) $\int \frac{3x+5}{x^2+4x+7}\,dx$ (ii) $\int \frac{x+1}{x^2-3x+1}\,dx$ (iii) $\int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx$ (iv) $\int \frac{5x-7}{\sqrt{3x-x^2-2}}\,dx$

Solution

(i) Let $I = \int \frac{3x+5}{x^2+4x+7}\,dx$

$$3x+5 = A\frac{d}{dx}(x^2+4x+7)+B$$

$$3x+5 = A(2x+4)+B$$

Comparing the coefficients of like terms, we get

$$2A = 3 \Rightarrow A = \frac{3}{2}; 4A+B = 5 \Rightarrow B = -1$$

$$I = \int \frac{\frac{3}{2}(2x+4)-1}{x^2+4x+7}\,dx$$

$$I = \frac{3}{2}\int \frac{2x+4}{x^2+4x+7}\,dx - \int \frac{1}{x^2+4x+7}\,dx$$

$$= \frac{3}{2}\log\left|x^2+4x+7\right| - \int \frac{1}{(x+2)^2+\left(\sqrt{3}\right)^2}\,dx$$

$$= \frac{3}{2}\log\left|x^2+4x+7\right| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right)+c$$

(ii) Let $I = \int \frac{x+1}{x^2-3x+1}\,dx$

$$x+1 = A\frac{d}{dx}(x^2-3x+1)+B$$

$$x + 1 = A(2x-3)+B$$

Comparing the coefficients of like terms, we get

$$2A = 1 \Rightarrow A = \frac{1}{2}; -3A + B = 1 \Rightarrow B = \frac{5}{2}$$

$$I = \int \frac{\frac{1}{2}(2x-3)+\frac{5}{2}}{x^2-3x+1}\,dx$$

$$I = \frac{1}{2}\int \frac{2x-3}{x^2-3x+1}\,dx + \frac{5}{2}\int \frac{1}{x^2-3x+1}\,dx$$

$$= \frac{1}{2}\log\left|x^2-3x+1\right| + \frac{5}{2}\int \frac{1}{\left(x-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\log\left|x^2-3x+1\right| + \frac{5}{2}\,\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\log\left|\frac{x-\frac{3}{2}-\frac{\sqrt{5}}{2}}{x-\frac{3}{2}+\frac{\sqrt{5}}{2}}\right| + c$$

$$= \frac{1}{2}\log\left|x^2-3x+1\right| + \frac{\sqrt{5}}{2}\log\left|\frac{2x-3-\sqrt{5}}{2x-3+\sqrt{5}}\right| + c$$

(iii) Let $I = \int \frac{2x+3}{\sqrt{x^2+x+1}}\,dx$

$$2x + 3 = A\frac{d}{dx}(x^2+x+1)+B$$

$$2x + 3 = A(2x + 1) + B$$

Comparing the coefficients of like terms, we get

$$2A = 2 \Rightarrow A = 1; \quad A + B = 3 \Rightarrow B = 2$$

$$I = \int \frac{(2x+1)+2}{\sqrt{x^2+x+1}}\,dx$$

$$I = \int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + 2\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$= 2\sqrt{x^2+x+1} + 2\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$= 2\sqrt{x^2+x+1} + 2\log\left|x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right| + c$$

Therefore, $I = 2\sqrt{x^2+x+1} + 2\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right| + c$

(iv) $\displaystyle\int \frac{5x-7}{\sqrt{3x-x^2-2}}dx$

$$5x-7 = A\frac{d}{dx}(3x-x^2-2)+B$$

$$5x-7 = A(3-2x)+B$$

Comparing the coefficients of like terms, we get

$$-2A = 5 \Rightarrow A = -\frac{5}{2}; 3A+B=-7 \Rightarrow B=\frac{1}{2}$$

$$I = \int \frac{\frac{-5}{2}(3-2x)+\frac{1}{2}}{\sqrt{3x-x^2+2}}dx$$

$$I = -\frac{5}{2}\int\frac{3-2x}{\sqrt{3x-x^2+2}}dx + \frac{1}{2}\int\frac{1}{\sqrt{3x-x^2+2}}dx$$

$$= \left(-\frac{5}{2}\right)2\sqrt{3x-x^2+2} + \frac{1}{2}\int\frac{1}{\sqrt{\left(\frac{\sqrt{17}}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$$

$$= -5\sqrt{3x-x^2+2} + \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{3}{2}}{\frac{\sqrt{17}}{2}}\right) + c$$

Thus, $I = -5\sqrt{3x-x^2+2} + \frac{1}{2}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right) + c$

EXERCISE 11.11

Integrate the following with respect to x :

(1) (i) $\dfrac{2x-3}{x^2+4x-12}$ (ii) $\dfrac{5x-2}{2+2x+x^2}$ (iii) $\dfrac{3x+1}{2x^2-2x+3}$

(2) (i) $\dfrac{2x+1}{\sqrt{9+4x-x^2}}$ (ii) $\dfrac{x+2}{\sqrt{x^2-1}}$ (iii) $\dfrac{2x+3}{\sqrt{x^2+4x+1}}$

Type IV

Integrals of the form $\int \sqrt{a^2 \pm x^2}\,dx, \int \sqrt{x^2 - a^2}\,dx$

Result 11.3

(1) $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

(2) $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + c$

(3) $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + c$

Proof :

(1) Let $I = \int \sqrt{a^2 - x^2}\,dx$

Take $u = \sqrt{a^2 - x^2}$ then $du = \frac{-2x}{2\sqrt{a^2 - x^2}}dx$

$dv = dx \Rightarrow v = x$

Applying Integration by parts, we get

$$\int u\,dv = uv - \int v\,du$$

$$\Rightarrow I = x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\sqrt{a^2 - x^2}}dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}dx$$

$$= x\sqrt{a^2 - x^2} - \int \left(\frac{a^2 - x^2}{\sqrt{a^2 - x^2}} + \frac{(-a^2)}{\sqrt{a^2 - x^2}}\right)dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}dx$$

$$= x\sqrt{a^2 - x^2} - I + a^2\int \frac{1}{\sqrt{a^2 - x^2}}dx$$

$$2I = x\sqrt{a^2 - x^2} + a^2\sin^{-1}\left(\frac{x}{a}\right)$$

Therefore, $I = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

Similarly we can prove other two results.

Note 11.3

The above problems can also be solved by substituting $x = a\sin\theta$

Examples 11.41

Evaluate the following :

(i) $\int \sqrt{4 - x^2}\,dx$ (ii) $\int \sqrt{25x^2 - 9}\,dx$ (iii) $\int \sqrt{x^2 + x + 1}\,dx$ (iv) $\int \sqrt{(x-3)(5-x)}\,dx$

Solution

(i) Let $I = \sqrt{4-x^2}dx$

$$= \int\sqrt{2^2-x^2}dx$$

$$= \frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\left(\frac{x}{2}\right)+c$$

Therefore, $I = \frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\left(\frac{x}{2}\right)+c$

(ii) Let $I = \int\sqrt{25x^2-9}dx$

$$= \int\sqrt{(5x)^2-3^2}dx$$

$$= \frac{1}{5}\left[\frac{5x}{2}\sqrt{(5x)^2-3^2}-\frac{3^2}{2}\log\left|5x+\sqrt{(5x)^2-3^2}\right|\right]+c$$

Therefore, $I = \frac{1}{5}\left[\frac{5x}{2}\sqrt{25x^2-9}-\frac{9}{2}\log\left|5x+\sqrt{25x^2-9}\right|\right]+c$

(iii) Let $I = \int\sqrt{x^2+x+1}\,dx$

$$= \int\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{x+\frac{1}{2}}{2}\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\log\left|\left[x+\frac{1}{2}+\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]\right|+c$$

Therefore, $I = \frac{2x+1}{4}\sqrt{x^2+x+1}+\frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|+c$

(iv) Let $I = \int\sqrt{(x-3)(5-x)}\,dx$

$$= \int\sqrt{8x-x^2-15}\,dx$$

$$= \int\sqrt{1^2-(x-4)^2}dx$$

$$= \frac{x-4}{2}\sqrt{1^2-(x-4)^2}+\frac{1}{2}\sin^{-1}\left(\frac{x-4}{1}\right)+c$$

Therefore, $I = \frac{x-4}{2}\sqrt{8x-x^2-15}+\frac{1}{2}\sin^{-1}(x-4)+c$

EXERCISE 11.12

Integrate the following functions with respect to x :

(1) (i) $\sqrt{x^2+2x+10}$ (ii) $\sqrt{x^2-2x-3}$ (iii) $\sqrt{(6-x)(x-4)}$

(2) (i) $\sqrt{9-(2x+5)^2}$ (ii) $\sqrt{81+(2x+1)^2}$ (iii) $\sqrt{(x+1)^2-4}$

BE4Z15

EXERCISE 11.13

Choose the correct or the most suitable answer from given four alternatives.

(1) If $\int f(x)dx = g(x)+c$, then $\int f(x)g'(x)dx$

(1) $\int (f(x))^2 dx$ (2) $\int f(x)g(x)dx$ (3) $\int f'(x)g(x)dx$ (4) $\int (g(x))^2 dx$

(2) If $\int \frac{3^{\frac{1}{x}}}{x^2} dx = k\,(3^{\frac{1}{x}})+c$, then the value of k is

(1) log 3 (2) $-\log 3$ (3) $-\frac{1}{\log 3}$ (4) $\frac{1}{\log 3}$

(3) If $\int f'(x)e^{x^2} dx = (x-1)e^{x^2}+c$, then $f(x)$ is

(1) $2x^3-\frac{x^2}{2}+x+c$ (2) $\frac{x^3}{2}+3x^2+4x+c$ (3) x^3+4x^2+6x+c (4) $\frac{2x^3}{3}-x^2+x+c$

(4) The gradient (slope) of a curve at any point (x, y) is $\frac{x^2-4}{x^2}$. If the curve passes through the point (2, 7), then the equation of the curve is

(1) $y = x+\frac{4}{x}+3$ (2) $y = x+\frac{4}{x}+4$ (3) $y = x^2+3x+4$ (4) $y = x^2-3x+6$

(5) $\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx$ is

(1) $\cot(xe^x)+c$ (2) $\sec(xe^x)+c$ (3) $\tan(xe^x)+c$ (4) $\cos(xe^x)+c$

(6) $\int \frac{\sqrt{\tan x}}{\sin 2x}dx$ is

(1) $\sqrt{\tan x}+c$ (2) $2\sqrt{\tan x}+c$ (3) $\frac{1}{2}\sqrt{\tan x}+c$ (4) $\frac{1}{4}\sqrt{\tan x}+c$

(7) $\int \sin^3 x\,dx$ is

(1) $\frac{-3}{4}\cos x-\frac{\cos 3x}{12}+c$ (2) $\frac{3}{4}\cos x+\frac{\cos 3x}{12}+c$

(3) $\frac{-3}{4}\cos x+\frac{\cos 3x}{12}+c$ (4) $\frac{-3}{4}\sin x-\frac{\sin 3x}{12}+c$

(8) $\int \frac{e^{6\log x}-e^{5\log x}}{e^{4\log x}-e^{3\log x}}dx$ is

(1) $x+c$ (2) $\frac{x^3}{3}+c$ (3) $\frac{3}{x^3}+c$ (4) $\frac{1}{x^2}+c$

(9) $\int \frac{\sec x}{\sqrt{\cos 2x}} dx$ is

(1) $\tan^{-1}(\sin x)+c$ (2) $2\sin^{-1}(\tan x)+c$ (3) $\tan^{-1}(\cos x)+c$ (4) $\sin^{-1}(\tan x)+c$

(10) $\int \tan^{-1}\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\, dx$ is

(1) x^2+c (2) $2x^2+c$ (3) $\frac{x^2}{2}+c$ (4) $-\frac{x^2}{2}+c$

(11) $\int 2^{3x+5} dx$ is

(1) $\frac{3(2^{3x+5})}{\log 2}+c$ (2) $\frac{2^{3x+5}}{2\log(3x+5)}+c$ (3) $\frac{2^{3x+5}}{2\log 3}+c$ (4) $\frac{2^{3x+5}}{3\log 2}+c$

(12) $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}\, dx$ is

(1) $\frac{1}{2}\sin 2x+c$ (2) $-\frac{1}{2}\sin 2x+c$ (3) $\frac{1}{2}\cos 2x+c$ (4) $-\frac{1}{2}\cos 2x+c$

(13) $\int \frac{e^x(x^2\tan^{-1}x+\tan^{-1}x+1)}{x^2+1}\, dx$ is

(1) $e^x\tan^{-1}(x+1)+c$ (2) $\tan^{-1}(e^x)+c$ (3) $e^x\frac{(\tan^{-1}x)^2}{2}+c$ (4) $e^x\tan^{-1}x+c$

(14) $\int \frac{x^2+\cos^2 x}{x^2+1}\operatorname{cosec}^2 x dx$ is

(1) $\cot x+\sin^{-1}x+c$ (2) $-\cot x+\tan^{-1}x+c$

(3) $-\tan x+\cot^{-1}x+c$ (4) $-\cot x-\tan^{-1}x+c$

(15) $\int x^2\cos x\, dx$ is

(1) $x^2\sin x+2x\cos x-2\sin x+c$ (2) $x^2\sin x-2x\cos x-2\sin x+c$

(3) $-x^2\sin x+2x\cos x+2\sin x+c$ (4) $-x^2\sin x-2x\cos x+2\sin x+c$

(16) $\int \sqrt{\frac{1-x}{1+x}}dx$ is

(1) $\sqrt{1-x^2}+\sin^{-1}x+c$ (2) $\sin^{-1}x-\sqrt{1-x^2}+c$

(3) $\log\left|x+\sqrt{1-x^2}\right|-\sqrt{1-x^2}+c$ (4) $\sqrt{1-x^2}+\log\left|x+\sqrt{1-x^2}\right|+c$

(17) $\int \frac{dx}{e^x-1}$ is

(1) $\log|e^x|-\log|e^x-1|+c$ (2) $\log|e^x|+\log|e^x-1|+c$

(3) $\log|e^x-1|-\log|e^x|+c$ (4) $\log|e^x+1|-\log|e^x|+c$

(18) $\int e^{-4x}\cos x\,dx$ is

(1) $\frac{e^{-4x}}{17}[4\cos x-\sin x]+c$ (2) $\frac{e^{-4x}}{17}[-4\cos x+\sin x]+c$

(3) $\frac{e^{-4x}}{17}[4\cos x+\sin x]+c$ (4) $\frac{e^{-4x}}{17}[-4\cos x-\sin x]+c$

(19) $\int \frac{\sec^2 x}{\tan^2 x-1}dx$

(1) $2\log\left|\frac{1-\tan x}{1+\tan x}\right|+c$ (2) $\log\left|\frac{1+\tan x}{1-\tan x}\right|+c$

(3) $\frac{1}{2}\log\left|\frac{\tan x+1}{\tan x-1}\right|+c$ (4) $\frac{1}{2}\log\left|\frac{\tan x-1}{\tan x+1}\right|+c$

(20) $\int e^{-7x}\sin 5x\,dx$ is

(1) $\frac{e^{-7x}}{74}[-7\sin 5x-5\cos 5x]+c$ (2) $\frac{e^{-7x}}{74}[7\sin 5x+5\cos 5x]+c$

(3) $\frac{e^{-7x}}{74}[7\sin 5x-5\cos 5x]+c$ (4) $\frac{e^{-7x}}{74}[-7\sin 5x+5\cos 5x]+c$

(21) $\int x^2 e^{\frac{x}{2}}dx$ is

(1) $x^2 e^{\frac{x}{2}}-4xe^{\frac{x}{2}}-8e^{\frac{x}{2}}+c$ (2) $2x^2 e^{\frac{x}{2}}-8xe^{\frac{x}{2}}-16e^{\frac{x}{2}}+c$

(3) $2x^2 e^{\frac{x}{2}}-8xe^{\frac{x}{2}}+16e^{\frac{x}{2}}+c$ (4) $x^2\frac{e^{\frac{x}{2}}}{2}-\frac{xe^{\frac{x}{2}}}{4}+\frac{e^{\frac{x}{2}}}{8}+c$

(22) $\int \frac{x+2}{\sqrt{x^2-1}}dx$ is

(1) $\sqrt{x^2-1}-2\log\left|x+\sqrt{x^2-1}\right|+c$ (2) $\sin^{-1}x-2\log\left|x+\sqrt{x^2-1}\right|+c$

(3) $2\log\left|x+\sqrt{x^2-1}\right|-\sin^{-1}x+c$ (4) $\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+c$

(23) $\int \frac{1}{x\sqrt{(\log x)^2-5}}dx$ is

(1) $\log\left|x+\sqrt{x^2-5}\right|+c$ (2) $\log\left|\log x+\sqrt{\log x-5}\right|+c$

(3) $\log\left|\log x+\sqrt{(\log x)^2-5}\right|+c$ (4) $\log\left|\log x-\sqrt{(\log x)^2-5}\right|+c$

(24) $\int \sin\sqrt{x}\,dx$ is

(1) $2\left(-\sqrt{x}\cos\sqrt{x}+\sin\sqrt{x}\right)+c$ (2) $2\left(-\sqrt{x}\cos\sqrt{x}-\sin\sqrt{x}\right)+c$

(3) $2\left(-\sqrt{x}\sin\sqrt{x}-\cos\sqrt{x}\right)+c$ (4) $2\left(-\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right)+c$

(25) $\int e^{\sqrt{x}}dx$ is

(1) $2\sqrt{x}(1-e^{\sqrt{x}})+c$ (2) $2\sqrt{x}(e^{\sqrt{x}}-1)+c$

(3) $2e^{\sqrt{x}}(1-\sqrt{x})+c$ (4) $2e^{\sqrt{x}}(\sqrt{x}-1)+c$

SUMMARY

Derivatives	Antiderivatives
$\frac{d}{dx}(c)=0,$ where c is a constant	$\int 0\,dx = c$, where c is a constant
$\frac{d}{dx}(kx)=k,$ where k is a constant	$\int k\,dx = kx + c$ where c is a constant
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^{n}$	$\int x^{n}dx = \frac{x^{n+1}}{n+1}+c,\quad n\neq -1$ (Power rule)
$\frac{d}{dx}\log x=\left(\frac{1}{x}\right)$	$\int \frac{1}{x}\,dx = \log\lvert x\rvert + c$
$\frac{d}{dx}(-\cos x)=\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\sin x)=\cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\tan x)=\sec^{2}x$	$\int \sec^{2}x\,dx = \tan x + c$
$\frac{d}{dx}(-\cot x)=\operatorname{cosec}^{2}x$	$\int \operatorname{cosec}^{2}x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x)=\sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + c$
$\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x$	$\int \operatorname{cosec} x\cot x\,dx = -\operatorname{cosec} x + c$
$\frac{d}{dx}(e^{x})=e^{x}$	$\int e^{x}dx = e^{x}+c$
$\frac{d}{dx}\left(\frac{a^{x}}{\log a}\right)=a^{x}$	$\int a^{x}dx = \frac{a^{x}}{\log a}+c$
$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^{2}}}$	$\int \frac{1}{\sqrt{1-x^{2}}}dx = \sin^{-1}x + c$
$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^{2}}$	$\int \frac{1}{1+x^{2}}dx = \tan^{-1}x + c$

(1) If k is any constant, then $\int kf(x)dx = k\int f(x)\,dx$

(2) $\int (f_1(x) \pm f_2(x))\,dx = \int f_1(x)dx \pm \int f_2(x)dx$

If $\int f(x)\,dx = g(x)+c$, then $\int f(ax+b)\,dx = \frac{1}{a}\,g(ax+b)+c$

(1)	$\int \tan x\,dx = \log\lvert\sec x\rvert + c$
(2)	$\int \cot x\,dx = \log\lvert\sin x\rvert + c$
(3)	$\int cosecx\,dx = \log\lvert\operatorname{cosec} x - \cot x\rvert + c$
(4)	$\int \sec x\,dx = \log\lvert\sec x + \tan x\rvert + c$

Bernoulli's formula for integration by Parts:

If u and v are functions of x, then the Bernoulli's rule is

$\int udv = uv - u'v_1 + u''v_2 + \ldots$

where $u', u'', u''', \ldots$ are successive derivatives of u and

$v, v_1, v_2, v_2, \ldots,$ are successive integrals of dv

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]+c$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]+c$$

$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert + c$	$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+c$
$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert + c$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left\lvert x+\sqrt{x^2-a^2}\right\rvert + c$	$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left\lvert x+\sqrt{x^2+a^2}\right\rvert + c$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$$

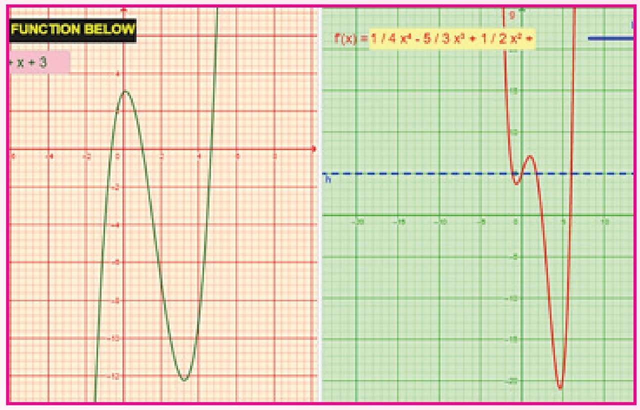
$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert+c$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert+c$$

ICT CORNER 11(a)

Integral Calculus

Expected Outcome

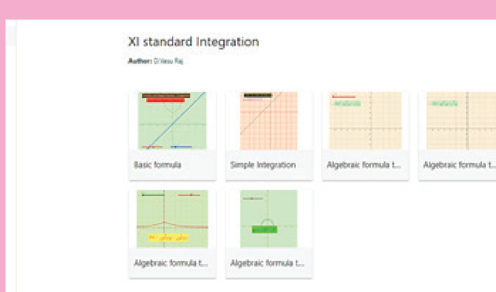


Step 1

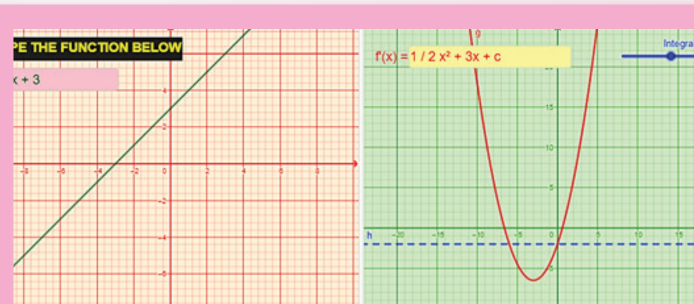
Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Integration" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Simple Integration". You can enter any function in the f(x)box. Graph of $f(x)$ appear on left side and the Integrated function will appear on right side. (Note: for x^5 enter x^5) Move the slider "integration constant" to change the constant value in integration.



Step1



Step2

Browse in the link:

XI standard Integration: https://ggbm.at/c63hdegc

B162_11_MAT_EM

ICT CORNER 11(b)

Integral Calculus

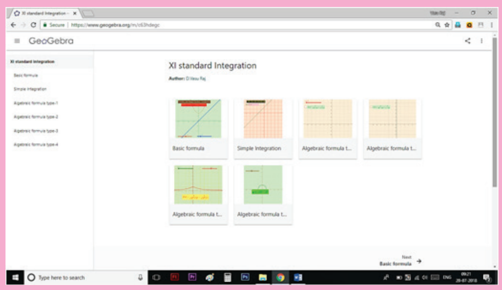
Expected Outcome

Step 1

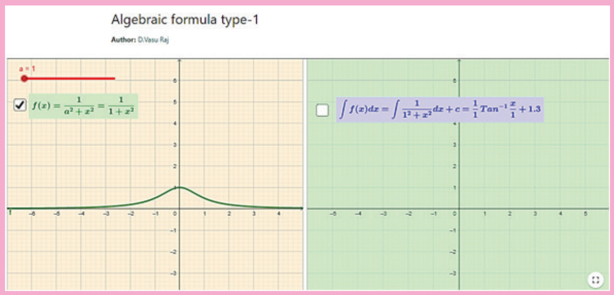
Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Integration" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Algebraic type-1". The graph of the function given on left side and the Integration of the function appear on right side. Click on both to see the graph. You can move the slider "a" to change the value. Algebraic types are grouped as 4 types open other three Algebraic types and observe.



Step1



Step2

Browse in the link:

XI standard Integration: https://ggbm.at/c63hdegc

B162_11_MAT_EM

Chapter 12 Introduction to Probability Theory

BEDV2S

The most important questions of life are, indeed, for the most part, really only problems of probability

Pierre - Simon Laplace

12.1 Introduction

Laplace
1749-1827

A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. The fundamental principles of probability theory were formulated by Pascal and Fermat for the first time. After an extensive research, Laplace published his monumental work in 1812, and laid the foundation to Probability theory. In statistics, the Bayesian interpretation of probability was developed mainly by Laplace.

The topic of probability is seen in many facets of the modern world. From its origin as a method of studying games, probability has involved in a powerful and widely applicable branch of mathematics. The uses of probability range from the determination of life insurance premium, to the prediction of election outcomes, the description of the behaviour of molecules in a gas. Its utility is one good reason why the study of probability has found in the way into a school textbook.
The interpretation of the word 'probability' involves synonyms such as chance, possible, probably, likely, odds, uncertainty, prevalence, risk, expectancy etc.

Our entire world is filled with uncertainty. We make decisions affected by uncertainty virtually every day. In order to measure uncertainty, we turn to a branch of mathematics called theory of probability. Probability is a measure of the likeliness that an event will occur.

Learning Objectives

On completion of this chapter, the students are expected to

- understand the classical theory of probability and axiomatic approach to probability.
- understand mutually exclusive, mutually inclusive and exhaustive events.
- understand the concepts of conditional probability and independent events.
- apply Bayes' theorem.
- apply probability theory in day-to-day life.

12.2 Basic definitions

Before we study the theory of probability, let us recollect the definition of certain terms already studied in earlier classes, which are frequently used.

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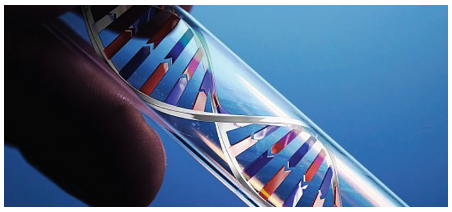
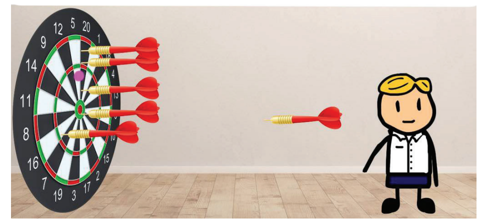
EXPERIMENT

Deterministic Experiment | **Random Experiment**



(Genetic determination)



(Hitting the target)

Definition 12.1

An **experiment** is defined as a process for which its result is well defined.

Definition 12.2

Deterministic experiment is an experiment whose outcomes can be predicted with certain, under ideal conditions.

Definition 12.3

A random experiment (or non-deterministic) is an experiment

(i) whose all possible outcomes are known in advance,

(ii) whose each outcome is not possible to predict in advance, and

(iii) can be repeated under identical conditions.

A die is 'rolled', a fair coin is 'tossed' are examples for random experiments.

Definition 12.4

A **simple event** (or elementary event or sample point) is the most basic possible outcome of a random experiment and it cannot be decomposed further.

Definition 12.5

A sample space is the set of all possible outcomes of a random experiment. Each point in sample space is an elementary event.

Illustration 12.1

(1) (i) If a die is rolled, then the sample space $S=\{1, 2, 3, 4, 5, 6\}$.

(ii) A coin is tossed, then the sample space $S=\{H, T\}$.

(2) (i) Suppose we toss a coin until a head is obtained. One cannot say in advance how many tosses will be required, and so the sample space.

$S=\{H, TH, TTH, TTTH, ...\}$ is an infinite set.

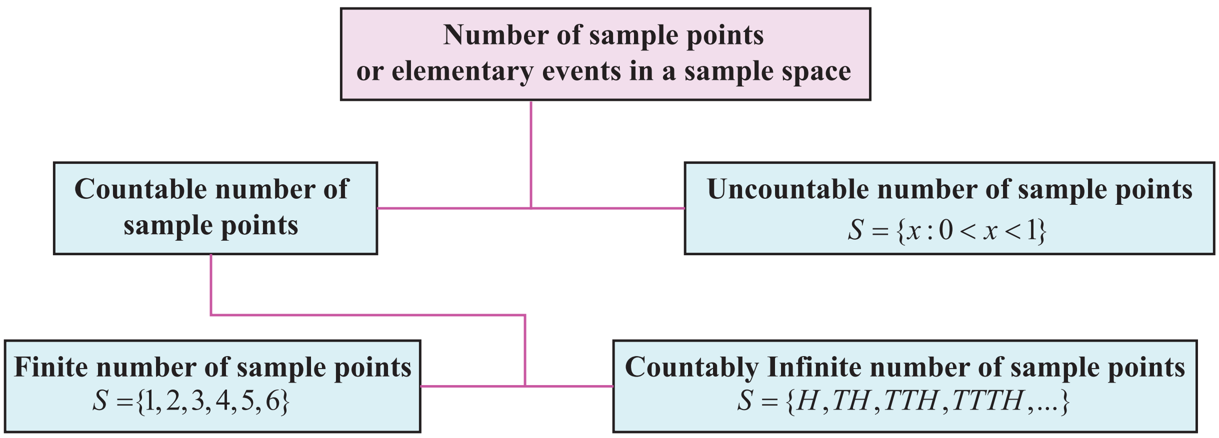
(ii) The sample space associated with the number of passengers waiting to buy train tickets in counters is $S=\{0,1,2,...\}$.

(3) (i) If the experiment consists of choosing a number randomly between 0 and 1, then the sample space is $S=\{x: 0<x<1\}$.

(ii) The sample space for the life length (t in hours) of a tube light is $S=\{t: 0<t<1000\}$.

From (2) and (3), one need to distinguish between two types of infinite sets, where one type is significantly 'larger' than the other. In particular, S in (2) is called **countably infinite**, while the S in (3) is called **uncountably infinite**. The fact that one can list the elements of a countably infinite set means that the set can be put in one-to-one correspondence with natural numbers $\mathbb{N}$. On the other hand, you cannot list the elements in uncountable set.

From the above example, one can understand that the sample space may consist of countable or uncountable number of elementary events.



12.3 Finite sample space

In this section we restrict our sample spaces that have at most a finite number of points.

Types of events

Let us now define some of the important types of events, which are used frequently in this chapter.

- Sure event or certain event
- Complementary event
- Mutually inclusive event
- Equally likely events
- Impossible event
- Mutually exclusive events
- Exhaustive events
- Independent events (defined after learning the concepts of probability)

Definition 12.6

When the sample space is finite, any subset of the sample space is an **event**. That is, all elements of the power set $\mathscr{P}(S)$ of the sample space are defined as **events**. An event is a collection of sample points or elementary events.

The sample space S is called **sure event or certain event.** The null set $\varnothing$ in S is called an **impossible event.**

Definition 12.7

For every event A, there corresponds another event $\overline{A}$ is called the **complementary event** to A. It is also called the event 'not A'.

Illustration 12.2

Suppose a sample space S is given by $S = \{1,2,3,4\}$.

Let the set of all possible subsets of S (the power set of S) be $\mathscr{P}(S)$.

$$\mathscr{P}(S) = \big\{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$$
$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\big\}$$

(i) All the elements of $\mathscr{P}(S)$ are events.

(ii) $\varnothing$ is an impossible event.

(iii) $\{1\},\{2\},\{3\},\{4\}$ are the simple events or elementary events.

(iv) $\{1, 2, 3, 4\}$ is a sure event or certain event.

Definition 12.8

Two events cannot occur simultaneously are mutually exclusive events. $A_1, A_2, A_3, \ldots, A_k$ are **mutually exclusive** means that, $A_i \cap A_j = \varnothing$, for $i \neq j$.

Definition 12.9

Two events are **mutually inclusive** when they can both occur simultaneously.

$A_1, A_2, A_3, \ldots, A_k$ are mutually inclusive means that, $A_i \cap A_j \neq \varnothing$, for $i \neq j$

Illustration 12.3

When we roll a die, the sample space $S = \{1,2,3,4,5,6\}$.

(i) Since $\{1, 3\} \cap \{2, 4, 5, 6\} = \varnothing$, the events $\{1,3\}$ and $\{2, 4,5,6\}$ are mutually exclusive events.

(ii) The events $\{1,6,\},\{2,3,5\}$ are mutually exclusive.

(iii) The events $\{2,3,5\},\{5,6\}$ are mutually inclusive, since $\{2, 3, 5\} \cap \{5, 6\} = \{5\} \neq \varnothing$

Definition 12.10

$A_1, A_2, A_3, \ldots, A_k$ are called **exhaustive events** if, $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Definition 12.11

$A_1, A_2, A_3, \ldots, A_k$ are called **mutually exclusive and exhaustive events** if,

(i) $A_i \cap A_j = \varnothing$, for $i \neq j$ (ii) $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = S$

Illustration 12.4

When a die is rolled, sample space $S = \{1,2,3,4,5,6\}$.

Some of the events are $\{2,3\},\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$.

(i) Since $\{2, 3\} \cup \{1, 3, 5\} \cup \{4, 6\} = \{1, 2, 3, 4, 5, 6\} = S$ (sample space), the events $\{2,3\},\{1,3,5\},\{4,6\}$ are exhaustive events.

(ii) Similarly $\{2,3\},\{4,6\}$ and $\{1,5\}$ are also exhaustive events.

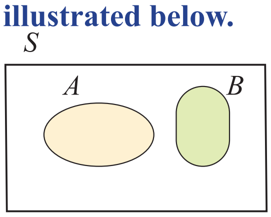
(iii) $\{1,3,5\},\{4,6\},\{6\}$ and $\{1,5\}$ are not exhaustive events.

(Since $\{1, 3, 5\} \cup \{4, 6\} \cup \{6\} \cup \{1, 5\} \neq S$)

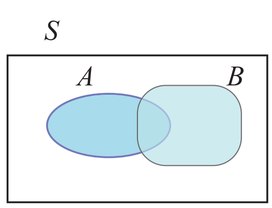
(iv) $\{2,3\},\{4,6\}$,and $\{1,5\}$ are mutually exclusive and exhaustive events, since

$\{2,3\} \cap \{4,6\} = \varnothing$, $\{2,3\} \cap \{1,5\} = \varnothing$, $\{4,6\} \cap \{1,5\} = \varnothing$ and $\{2, 3\} \cup \{4, 6\} \cup \{1, 5\} = S$

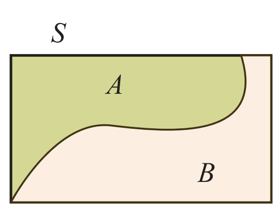
Types of events associated with sample space are easy to visualize in terms of Venn diagrams, as illustrated below.



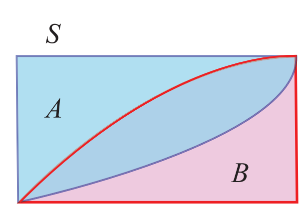
A and B are Mutually exclusive



A and B are Mutually inclusive



A and B are Mutually exclusive and exhaustive



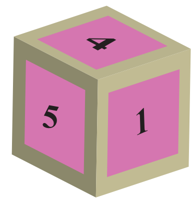
A and B are Mutually inclusive and exhaustive

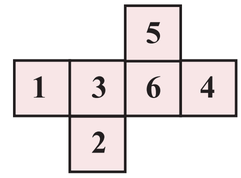
Definition 12.12

The events having the same chance of occurrences are called **equally likely** events.

Example for equally likely events: Suppose a fair die is rolled.

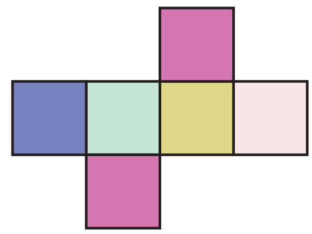
Number on the face	1	2	3	4	5	6
Chance of occurrence	1	1	1	1	1	1

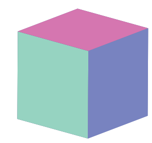




Example for not equally likely events: A colour die is shown in figure is rolled.

Colour on the face					
Chance of occurrence	1	1	1	2	1





Similarly, suppose if we toss a coin, the events of getting a head or a tail are equally likely.

Methods to find sample space

Illustration 12.5

Two coins are tossed, the sample space is

(i) $S=\{H,T\}\times\{H,T\}=\{(H,H),(H,T),(T,H),(T,T)\}$ or $\{HH,HT,TH,TT\}$

(ii) If a coin is tossed and a die is rolled simultaneously, then the sample space is

$$S=\{H,T\}\times\{1,2,3,4,5,6\} = \{H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6\} \text{ or}$$

$$S=\{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

Also one can interchange the order of outcomes of coin and die. The following table gives the sample spaces for some random experiments.

Random Experiment	Total Number of Outcomes	Sample space
Tossing a fair coin	$2^1=2$	$\{H, T\}$
Tossing two coins	$2^2=4$	$\{HH, HT, TH, TT\}$
Tossing three coins	$2^3=8$	$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
Rolling fair die	$6^1=6$	$\{1, 2,3, 4, 5,6\}$
Rolling Two dice or single die two times.	$6^2=36$	$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$ $(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$ $(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$ $(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$ $(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$ $(6,1), (6,2),(6,3), (6,4), (6,5), (6,6)\}$
Drawing a card from a pack of 52 playing cards	$52^1=52$	Heart ♥ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Diamond ♦ A 2 3 4 5 6 7 8 9 10 J Q K Red in colour Spade ♠ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour Club ♣ A 2 3 4 5 6 7 8 9 10 J Q K Black in colour

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Notations

Let A and B be two events.

(i) $A \cup B$ stands for the occurrence of A or B or both.

(ii) $A \cap B$ stands for the simultaneous occurrence of A and B. $A \cap B$ can also be written as AB

(iii) $\overline{A}$ or A' or A^c stands for non-occurrence of A

(iv) $(A \cap \overline{B})$ stands for the occurrence of only A.

Priori : Knowledge which precedes from theoretical deduction or making assumption. Not from experience or observation

12.4 Probability

12.4.1 Classical definition (A priori) of probability (Bernoulli's principle of equally likely)

Earlier classes we have studied the frequency (A posteriori) definition of probability and the problems were solved. Now let us learn the fundamentals of the axiomatic approach to probability theory.

Posteriori : Knowledge which precedes from experience or observation.

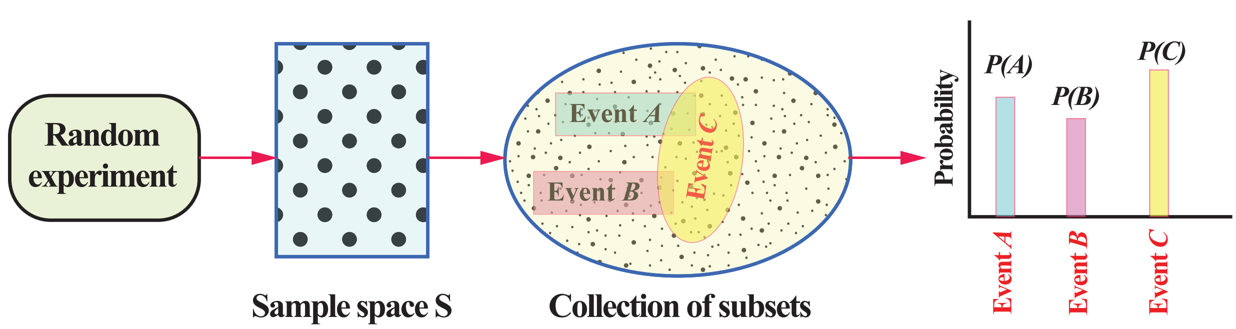
The basic assumption of underlying the classical theory is that the outcomes of a random experiment are equally likely. If there are n exhaustive, mutually exclusive and equally likely outcomes of an experiment and m of them are favorable to an event A, then the mathematical probability of A is defined as the ratio $\frac{m}{n}$. In other words, $P(A) = \frac{m}{n}$.

Definition 12.13

Let S be the sample space associated with a random experiment and A be an event. Let $n(S)$ and $n(A)$ be the number of elements of S and A respectively. Then the probability of the event A is defined as

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive number of cases in } S}$$

Every probabilistic model involves an underlying process is shown in the following figure.



The classical definition of probability is limited in its application only to situations where there are a finite number of possible outcomes. It mainly considered discrete events and its methods were mainly combinatorial. This renders it inapplicable to some important random experiments, such as 'tossing a coin until a head appears' which give rise to the possibility of infinite set of outcomes. Another limitation of the classical definition was the condition that each possible outcome is 'equally likely'.

These types of limitations in the classical definition of probability led to the evolution of the modern definition of probability which is based on the concept of sets. It is known an axiomatic approach.

The foundations of the Modern Probability theory were laid by Andrey Nikolayevich Kolmogorov, a Russian mathematician who combined the notion of sample space introduced by Richard von Mises, and measure theory and presented his axiomatic system for probability theory in 1933. We introduce the axiomatic approach proposed by A.N. Kolmogorov. Based on this, it is possible to construct a logically perfect structure of the modern theory of probability theory. The classical theory of probability is a particular case of axiomatic probability. The axioms are a set of rules, which can be used to prove theorems of probability.


A.N. Kolmogorov

12.4.2 Axiomatic approach to Probability

Axioms of probability

Let S be a finite sample space, let $\mathscr{P}(S)$ be the class of events, and let P be a real valued function defined on $\mathscr{P}(S)$. Then $P(A)$ is called probability function of the event A, when the following axioms are hold:

[P_1] For any event A, $P(A) \geq 0$ (Non-negativity axiom)

[P_2] For any two mutually exclusive events $P(A \cup B) = P(A) + P(B)$ (Additivity axiom)

[P_3] For the certain event $P(S) = 1$ (Normalization axiom)

Note 12.1

(i) $0 \leq P(A) \leq 1$

(ii) If $A_1, A_2, A_3, ..., A_n$ are mutually exclusive events in a sample space S, then
$$P(A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \cdots + P(A_n)$$

Theorems on finite probability spaces (without proof)

When the outcomes are equally likely Theorem 12.1 is applicable, else Theorem 12.2 is applicable.

Theorem 12.1

Let S be a sample space and for any subset A of S, let $P(A) = \dfrac{n(A)}{n(S)}$. Then $P(A)$ satisfies axioms of probability $[P_1], [P_2]$, and $[P_3]$.

Theorem12.2

Let S be a finite sample space say $S = \{a_1, a_2, a_3, ..., a_n\}$. A finite probability space is obtained by assigning to each point a_i in S a real number p_i, is called the probability of a_i, satisfying the following properties:

(i) Each $p_i \geq 0$. (ii) The sum of the p_i is 1, that is, $\sum p_i = p_1 + p_2 + p_3 + \cdots + p_1 = 1$.

If the probability $P(A)$, of an event A is defined as the sum of the probabilities of the points in A, then the function $P(A)$ satisfies the axioms of probability $[P_1], [P_2]$, and $[P_3]$.

Note: Sometimes the points in a finite sample space and their assigned probabilities are given in the form of a table as follows:

Outcome	a_1	a_2	a_3	$\ldots$	a_n
Probability	p_1	p_2	p_3	$\ldots$	p_n

Here is an illustration of how to construct a probability law starting from some common sense assumptions about a model.

Illustration 12.6

(1) Let $S=\{1,2,3\}$. Suppose $\mathscr{P}(S)$ is the power set of S, and $P(A)=\dfrac{n(A)}{n(S)}$.

Then $P(\{1\})=\dfrac{1}{3}$, $P(\{2\})=\dfrac{1}{3}$, and $P(\{3\})=\dfrac{1}{3}$,

satisfies axioms of probability $[P_1],[P_2]$, and $[P_3]$. Here all the outcomes are equally likely.

(2) Let $S=\{1,2,3\}$. Suppose $\mathscr{P}(S)$ is the power set of S,

If the probability $P(A)$, of an event A of S is defined as the sum of the probabilities of the points in A,

then $P(\{1\})=\dfrac{1}{2}$, $P(\{2\})=\dfrac{1}{4}$, $P(\{3\})=\dfrac{1}{4}$,

satisfy the axioms of probability $[P_1],[P_2]$, and $[P_3]$.

(3) Let $S=\{1,2,3\}$ and $\mathscr{P}(S)$ is the power set of S. If the probability $P(A)$, of an event A of S is defined as the sum of the probabilities of the points in A,

then $P(\{1\})=0$, $P(\{2\})=\dfrac{1}{\sqrt{2}}$, and $P(\{3\})=1-\dfrac{1}{\sqrt{2}}$,

satisfy the above axioms $[P_1],[P_2]$, and $[P_3]$.

In (2) and (3), the outcomes are not equally likely.

Note 12.2

Irrational numbers also can act as probabilities.

Classroom Activity: Each student to flip a coin 10 times,

Calculate: $p=\dfrac{\text{Number of times heads occur}}{10}$

Find the cumulative ratio of heads to tosses. As number of tosses increases $p\to\dfrac{1}{2}$

Example 12.1

If an experiment has exactly the three possible mutually exclusive outcomes A, B, and C, check in each case whether the assignment of probability is permissible.

(i) $P(A)=\dfrac{4}{7}$, $P(B)=\dfrac{1}{7}$, $P(C)=\dfrac{2}{7}$.

(ii) $P(A)=\dfrac{2}{5}$, $P(B)=\dfrac{1}{5}$, $P(C)=\dfrac{3}{5}$.

(iii) $P(A)=0.3$, $P(B)=0.9$, $P(C)=-0.2$.

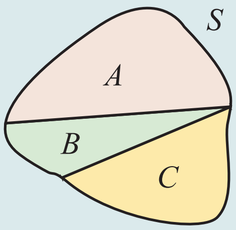
(iv) $P(A)=\dfrac{1}{\sqrt{3}}$, $P(B)=1-\dfrac{1}{\sqrt{3}}$, $P(C)=0$.

(v) $P(A)=0.421$, $P(B)=0.527$ $P(C)=0.042$.

Solution

Since the experiment has exactly the three possible mutually exclusive outcomes A, B and C, they must be exhaustive events.

$$\Rightarrow S = A \cup B \cup C$$



Therefore, by axioms of probability

$P(A) \geq 0,\quad P(B) \geq 0,\quad P(C) \geq 0$ and

$P(A \cup B \cup C) = P(A) + P(B) + P(C) = P(S) = 1$

(i) Given that $P(A) = \frac{4}{7} \geq 0$, $\quad P(B) = \frac{1}{7} \geq 0$, and $\quad P(C) = \frac{2}{7} \geq 0$

Also $P(S) = P(A) + P(B) + P(C) = \frac{4}{7} + \frac{1}{7} + \frac{2}{7} = 1$

Therefore the assignment of probability is permissible.

(ii) Given that $P(A) = \frac{2}{5} \geq 0$, $\quad P(B) = \frac{1}{5} \geq 0$, and $\quad P(C) = \frac{3}{5} \geq 0$

But $P(S) = P(A) + P(B) + P(C) = \frac{2}{5} + \frac{1}{5} + \frac{3}{5} = \frac{6}{5} > 1$

Therefore the assignment is not permissible.

(iii) Since $P(C) = -0.2$ is negative, the assignment is not permissible.

(iv) The assignment is permissible because

$P(A) = \frac{1}{\sqrt{3}} \geq 0$, $\quad P(B) = 1 - \frac{1}{\sqrt{3}} \geq 0$, and $P(C) = 0 \geq 0$

$P(S) = P(A) + P(B) + P(C) = \frac{1}{\sqrt{3}} + 1 - \frac{1}{\sqrt{3}} + 0 = 1.$

(v) Even though $P(A) = 0.421 \geq 0$, $P(B) = 0.527 \geq 0$, and $P(C) = 0.042 \geq 0$, the sum of the probability

$P(S) = P(A) + P(B) + P(C) = 0.421 + 0.527 + 0.042 = 0.990 < 1$.

Therefore, the assignment is not permissible.

Example 12.2

An integer is chosen at random from the first ten positive integers. Find the probability that it is (i) an even number (ii) multiple of three.

Solution

The sample space is

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\},\quad n(S) = 10$$

Let A be the event of choosing an even number and

B be the event of choosing an integer multiple of three.

$A = \{2, 4, 6, 8, 10\}, \quad n(A) = 5,$

$B = \{3, 6, 9\}, \quad n(B) = 3$

$$P\text{ (choosing an even integer)} = P(A) = \frac{n(A)}{n(S)} = \frac{5}{10} = \frac{1}{2}.$$

$$P\text{ (choosing an integer multiple of three)} = P(B) = \frac{n(B)}{n(S)} = \frac{3}{10}.$$

Example 12.3

Three coins are tossed simultaneously, what is the probability of getting (i) exactly one head (ii) at least one head (iii) at most one head?

Solution:

Notice that three coins are tossed simultaneously = one coin is tossed three times.

The sample space $S = \{H,T\} \times \{H,T\} \times \{H,T\}$

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},\ n(S) = 8$$

Let A be the event of getting exactly one head, B be the event of getting atleast one head and C be the event of getting at most one head.

$$A = \{HTT, THT, TTH\};\ n(A) = 3$$

$$B = \{HTT, THT, TTH, HHT, HTH, THH, HHH\};\quad n(B) = 7$$

$$C = \{TTT, HTT, THT, TTH\};\quad n(C) = 4\,.$$

Therefore the required probabilities are

(i) $\quad P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{8}$

(ii) $\quad P(B) = \dfrac{n(B)}{n(S)} = \dfrac{7}{8}$

(iii) $\quad P(C) = \dfrac{n(C)}{n(S)} = \dfrac{4}{8} = \dfrac{1}{2}.$

Note 12.3

When the number of elements in a sample space is considerably small we can solve by finger-counting the elements in the events. But when the number of elements is too large to count then combinatorics helps us to solve the problems.

For the following problem, combinatorics is used to find the number of elements in the sample space and the events.

Example 12.4

Suppose ten coins are tossed. Find the probability to get (i) exactly two heads (ii) at most two heads (iii) at least two heads

Solution

Ten coins are tossed simultaneously one time = one coin is tossed 10 times

Let S the sample space,

That is $\quad S = \overbrace{\{H,T\} \times \{H,T\} \times \{H,T\} \times \cdots \times \{H,T\}}^{10\text{ times}}$

Let A be the event of getting exactly two heads,

B be the event of getting at most two heads, and

C be the event of getting at least two heads.

When ten coins are tossed, the number of elements in sample space is $2^n = 2^{10} = 1024$

$$n(S) = 1024$$

$$n(A) = {}^{10}C_2 = 45$$

$$n(B) = {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 = 1 + 10 + 45 = 56$$

$$n(C) = {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 + \cdots + {}^{10}C_{10}$$

$$= n(S) - ({}^{10}C_0 + {}^{10}C_1) = 1024 - 11 = 1013$$

The required probabilities are

(i) $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{45}{1024}$

(ii) $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{56}{1024} = \dfrac{7}{128}$

(iii) $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{1013}{1024}$.

Example 12.5

Suppose a fair die is rolled. Find the probability of getting

(i) an even number (ii) multiple of three.

Solution

Let S be the sample space,

A be the event of getting an even number,

B be the event of getting multiple of three.

Therefore,

$S = \{1, 2, 3, 4, 5, 6\}$. $\Rightarrow n(S) = 6$

$A = \{2, 4, 6\}$ $\Rightarrow n(A) = 3$

$B = \{3, 6\}$ $\Rightarrow n(B) = 2$

The required probabilities are

(i) P (getting an even number) $= P(A) = \dfrac{n(A)}{n(S)} = \dfrac{3}{6} = \dfrac{1}{2}$

(ii) P (getting multiple of three) $= P(B) = \dfrac{n(B)}{n(S)} = \dfrac{2}{6} = \dfrac{1}{3}$.

Example 12.6

When a pair of fair dice is rolled, what are the probabilities of getting the sum

(i) 7 (ii) 7 or 9 (iii) 7 or 12?

Solution

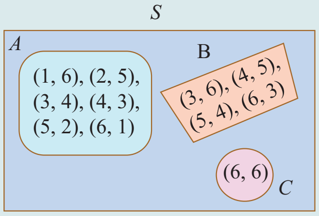
The sample space $S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$

$$\begin{aligned} S = \{&(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),\\ &(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),\\ &(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),\\ &(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),\\ &(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),\\ &(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} \end{aligned}$$

Number of possible outcomes $= 6^2 = 36 = n(S)$

Let A be the event of getting sum 7, B be the event of getting the sum 9 and C be the event of getting sum 12. Then

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} \quad \Rightarrow n(A) = 6$$
$$B = \{(3,6), (4,5), (5,4), (6,3)\} \quad \Rightarrow n(B) = 4$$
$$C = \{(6,6)\} \quad \Rightarrow n(C) = 1$$



(i) P (getting sum 7) $= P(A)$

$$= \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(ii) P (getting sum 7 or 9) $= P(A \text{ or } B) = P(A \cup B)$

$$= P(A) + P(B)$$

(Since A and B are mutually exclusive that is, $A \cap B = \emptyset$)

$$= \frac{n(A)}{n(S)} + \frac{n(B)}{n(S)} = \frac{6}{36} + \frac{4}{36} = \frac{5}{18}$$

(iii) P (getting sum 7 or 12) $= P(A \text{ or } C) = P(A \cup C)$

$$= P(A) + P(C) \quad \text{(since } A \text{ and } C \text{ are mutually exclusive)}$$

$$= \frac{n(A)}{n(S)} + \frac{n(C)}{n(S)} = \frac{6}{36} + \frac{1}{36} = \frac{7}{36}.$$

Example 12.7

Three candidates X, Y, and Z are going to play in a chess competition to win FIDE (World Chess Federation) cup this year. X is thrice as likely to win as Y and Y is twice as likely as to win Z. Find the respective probability of X, Y and Z to win the cup.

Solution

Let A, B, C be the event of winning FIDE cup respectively by X, Y, and Z this year.

Given that X is thrice as likely to win as Y.

$$A : B \; : : \; 3 : 1. \qquad (1)$$

Y is twice as likely as to win Z

$$B : C : : \; 2 : 1 \qquad (2)$$

From (1) and (2)

$$A : B : C : : \; 6 : 2 : 1$$

$A = 6k$, $B = 2k$, $C = k$, where k is proportional constant.

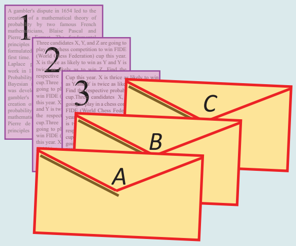
Probability to win the cup by X is $\quad P(A) = \frac{6k}{9k} = \frac{2}{3}$

Probability to win the cup by Y is $\quad P(B) = \frac{2k}{9k} = \frac{2}{9}$ and

Probability to win the cup by Z is $\quad P(C) = \frac{k}{9k} = \frac{1}{9}$.

Example 12.8

Three letters are written to three different persons and addresses on three envelopes are also written. Without looking at the addresses, what is the probability that (i) exactly one letter goes to the right envelopes (ii) none of the letters go into the right envelopes?



Solution

Let A, B, and C denote the envelopes and 1, 2, and 3 denote the corresponding letters.

The different combination of letters put into the envelopes are shown in the table.

Let c_i denote the outcomes of the events.

Let X be the event of putting the letters into the exactly only one right envelopes.

Let Y be the event of putting none of the letters into the right envelope.

Outcomes

Envelope	c_1	c_2	c_3	c_4	c_5	c_6
A	1	1	2	2	3	3
B	2	3	1	3	1	2
C	3	2	3	1	2	1

$S = \{c_1, c_2, c_3, c_4, c_5, c_6\}$, $n(S) = 6$

$$X = \{c_2, c_3, c_6\}, \qquad n(X) = 3$$

$$Y = \{c_4, c_5\} \qquad n(Y) = 2$$

$$P(X) = \frac{3}{6} = \frac{1}{2} \qquad P(Y) = \frac{2}{6} = \frac{1}{3}.$$

Example 12.9

Let the matrix $M = \begin{bmatrix} x & y \\ z & 1 \end{bmatrix}$. If x, y and z are chosen at random from the set $\{1, 2, 3\}$, and repetition is allowed $(i.e., x = y = z)$, what is the probability that the given matrix M is a singular matrix?

Solution

If the given matrix M is singular, then

$$\begin{vmatrix} x & y \\ z & 1 \end{vmatrix} = 0.$$

That is, $x - yz = 0$.

Hence the possible ways of selecting (x, y, z) are

$\{(1,1,1), (2,1,2), (2,2,1), (3,1,3), (3,3,1)\} = A(say)$

The number of favourable cases $\quad n(A) = 5$

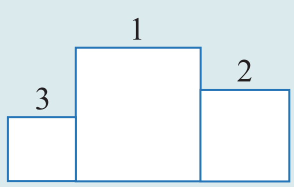
The total number of cases are $\quad n(S) = 3^3 = 27$

The probability of the given matrix is a singular matrix is

$$p = \frac{n(A)}{n(S)} = \frac{5}{27}.$$

Example 12.10

For a sports meet, a winners' stand comprising of three wooden blocks is in the form as shown in figure. There are six different colours available to choose from and three of the wooden blocks is to be painted such that no two of them has the same colour. Find the probability that the smallest block is to be painted in red, where red is one of the six colours.



Solution

Let S be the sample space and A be the event that the smallest block is to be painted in red.

$$n(S) = 6P_3 = 6 \times 5 \times 4 = 120$$

$$n(A) = 5 \times 4 = 20$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{20}{120} = \frac{1}{6}.$$

	6	6	3
$n(S)$	6	5	4
$n(A)$	5	4	Red

12.4.3 ODDS

The word **odds** is frequently used in probability and statistics. Odds relate the chances in favour of an event A to the chances against it. Suppose 'a' represents the number of ways that an event can occur and 'b' represents the number of ways that the event can fail to occur.

The odds of an event A are $a : b$ in favour of an event and

$$P(A) = \frac{a}{a+b}.$$

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Further, it may be noted that the odds are $a : b$ in favour of an event is the same as to say that the odds are $b : a$ against the event.

If the probability of an event is p, then the odds in favour of its occurrence are p to $(1-p)$ and the odds against its occurrence are $(1-p)$ to p.

Illustration 12.7

(i) Suppose a die is rolled.

Let S be the sample space and A be the event of getting 5.

BEHR4F

$n(S)=6$, $n(A)=1$ and $n(\bar{A})=5$.

It can also be interpreted as

Odds in favour of A is $1:5$ or $\frac{1}{5}$, odds against A is $5:1$ or $\frac{5}{1}$,

and $$P(A)=\frac{n(A)}{n(A)+n(\bar{A})}=\frac{1}{5+1}=\frac{1}{6} = \frac{n(A)}{n(S)}.$$

(ii) Suppose B is an event such that odds in **favour** of B is 3:5, then $P(B)=\frac{3}{8}$

(iii) Suppose C is an event such that odds **against** C is 4:11, then $P(C)=\frac{11}{15}$.

Example 12.11

A man has 2 ten rupee notes, 4 hundred rupee notes and 6 five hundred rupee notes in his pocket. If 2 notes are taken at random, what are the odds in favour of both notes being of hundred rupee denomination and also its probability?

Solution

Let S be the sample space and A be the event of taking 2 hundred rupee note.

Therefore, $n(S)=12c_2=66$, $n(A)=4c_2=6$ and $n(\bar{A})=66-6=60$

Therefore, odds in favour of A is 6: 60

That is, odds in favour of A is 1: 10, and $P(A)=\frac{1}{11}$.

EXERCISE 12.1

(1) An experiment has the four possible mutually exclusive and exhaustive outcomes A, B, C, and D. Check whether the following assignments of probability are permissible.

(i) $P(A) = 0.15$, $P(B) = 0.30$, $P(C) = 0.43$, $P(D)= 0.12$

(ii) $P(A) = 0.22$, $P(B) = 0.38$, $P(C) = 0.16$, $P(D) = 0.34$

(iii) $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{5}$, $P(C) = -\frac{1}{5}$, $P(D) = \frac{1}{5}$

(2) If two coins are tossed simultaneously, then find the probability of getting

(i) one head and one tail (ii) atmost two tails

(3) Five mangoes and 4 apples are in a box. If two fruits are chosen at random, find the probability that (i) one is a mango and the other is an apple (ii) both are of the same variety.

(4) What is the chance that
(i) non-leap year (ii) leap year should have fifty three Sundays?

(5) Eight coins are tossed once, find the probability of getting
(i) exactly two tails (ii) atleast two tails (iii) atmost two tails

(6) An integer is chosen at random from the first 100 positive integers. What is the probability that the integer chosen is a prime or multiple of 8?

(7) A bag contains 7 red and 4 black balls, 3 balls are drawn at random.
Find the probability that (i) all are red (ii) one red and 2 black.

(8) A single card is drawn from a pack of 52 cards. What is the probability that
(i) the card is an ace or a king (ii) the card will be 6 or smaller
(iii) the card is either a queen or 9?

(9) A cricket club has 16 members, of whom only 5 can bowl. What is the probability that in a team of 11 members at least 3 bowlers are selected?

(10) (i) The odds that the event A occurs is 5 to 7, find $P(A)$.
(ii) Suppose $P(B) = \frac{2}{5}$. Express the odds that the event B occurs.

12.5 Some basic Theorems on Probability

The problems solved in the last sections are related to mutually exclusive events. So we have used the formula $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$. But when the events are mutually inclusive, the additivity axioms counts $(A \cap B)$ twice. We have a separate formula for the events when they are mutually inclusive.

In the development of probability theory, all the results are derived directly or indirectly using only the axioms of probability. Here we derive some of the basic important theorems on probability.

Theorem 12.3

The probability of the impossible event is zero. That is,

$$\boxed{P(\varnothing) = 0}$$

Proof

Impossible event contains no sample point.

$$\begin{aligned} \text{Therefore, } S \cup \varnothing &= S \\ P(S \cup \varnothing) &= P(S) \\ P(S) + P(\varnothing) &= P(S) \quad \text{(since } S \text{ and } \varnothing \text{ are mutually exclusive)} \\ P(\varnothing) &= 0 \end{aligned}$$

Example 12.12

Find the probability of getting the number 7, when a usual die is rolled.

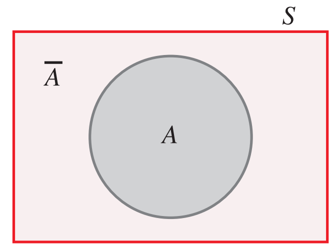
Solution

The event of getting 7 is an impossible event. Therefore, $P(\text{getting } 7) = 0$

Theorem 12.4

If $\overline{A}$ is the complementary event of A, then

$$\boxed{P(\overline{A}) = 1 - P(A)}$$



Proof

Let S be a sample space, we have

$$A \cup \overline{A} = S$$

$$P(A \cup \overline{A}) = P(S)$$

$$P(A) + P(\overline{A}) = P(S) \quad \text{(since } A \text{ and } \overline{A} \text{ are mutually exclusive)}$$

$$= 1$$

$$P\left(\overline{A}\right) = 1 - P\left(A\right) \text{ or } P\left(A\right) = 1 - P\left(\overline{A}\right)$$

Example 12.13

Nine coins are tossed once, find the probability to get at least two heads.

Solution

Let S be the sample space and A be the event of getting at least two heads.

Therefore, the event $\overline{A}$ denotes, getting at most one head.

$$n(S) = 2^9 = 512, \quad n(\overline{A}) = 9C_0 + 9C_1 = 1 + 9 = 10$$

It is easier to calculate $\sum_{r=0}^{1} 9C_r$ than $\sum_{r=2}^{9} 9C_r$

$$P\left(\overline{A}\right) = \frac{10}{512} = \frac{5}{256}$$

$$P\left(A\right) = 1 - P\left(\overline{A}\right) = 1 - \frac{5}{256} = \frac{251}{256}$$

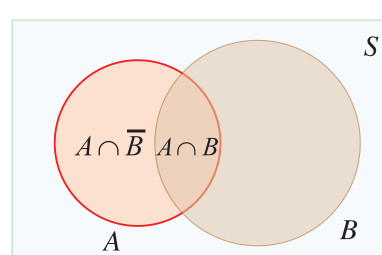
Theorem 12.5

If A and B are any two events and $\overline{B}$ is the complementary events of B, then

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

Proof

Clearly from the figure,



$$(A \cap \overline{B}) \cup (A \cap B) = A$$

$$P\left[(A \cap \overline{B}) \cup (A \cap B)\right] = P(A)$$

$$P(A \cap \overline{B}) + P(A \cap B) = P(A)$$

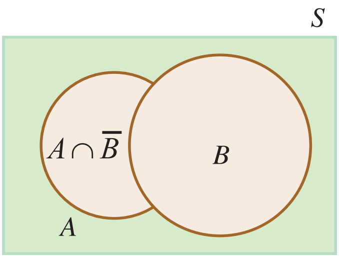
(since $(A \cap \overline{B})$ and $(A \cap B)$ are mutually exclusive)

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

Theorem 12.6 (Addition theorem on probability)

If A and B are any two events, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



Proof

From the diagram,

$$A \cup B = (A \cap \overline{B}) \cup B$$

$$P(A \cup B) = P\left[(A \cap \overline{B}) \cup B\right]$$

$$= P(A \cap \overline{B}) + P(B) \qquad \text{(since } (A \cap \overline{B}) \text{ and } B \text{ are mutually exclusive)}$$

$$= [P(A) - P(A \cap B)] + P(B)$$

Therefore, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Note 12.4

The above theorem can be extended to any 3 events.

(i) $P(A \cup B \cup C) = \{P(A) + P(B) + P(C)\}$

$- \{P(A \cap B) + P(B \cap C) + P(C \cap A)\} + P(A \cap B \cap C)\}$

(ii) $P(A \cup B \cup C) = 1 - P(\overline{A \cup B \cup C}) = 1 - P(\overline{A} \cap \overline{B} \cap \overline{C})$

Example 12.14

Given that $P(A) = 0.52$, $P(B) = 0.43$, and $P(A \cap B) = 0.24$, find

(i) $P(A \cap \overline{B})$ (ii) $P(A \cup B)$ (iii) $P(\overline{A} \cap \overline{B})$ (iv) $P(\overline{A} \cup \overline{B})$.

Solution

(i) $P(A \cap \overline{B}) = P(A) - P(A \cap B)$

$= 0.52 - 0.24 = 0.28$

$P(A \cap \overline{B}) = 0.28.$

(ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.52 + 0.43 - 0.24$

$P(A \cup B) = 0.71.$

(iii) $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$ (By de Morgan's law)

$= 1 - P(A \cup B)$

$= 1 - 0.71 = 0.29.$

(iv) $P(\overline{A} \cup \overline{B}) = P(\overline{A \cap B})$ (By de Morgan's law)

$= 1 - P(A \cap B) = 1 - 0.24$

$= 0.76.$

Example 12.15

The probability that a girl, preparing for competitive examination will get a State Government service is 0.12, the probability that she will get a Central Government job is 0.25, and the probability that she will get both is 0.07. Find the probability that (*i*) she will get atleast one of the two jobs (*ii*) she will get only one of the two jobs.

Solution

Let I be the event of getting State Government service and C be the event of getting Central Government job.

Given that $P(I) = 0.12$, $P(C) = 0.25$, and $P(I \cap C) = 0.07$

(*i*) P (at least one of the two jobs) $= P(I \text{ or } C) = P(I \cup C)$

$$= P(I) + P(C) - P(I \cap C)$$

$$= 0.12 + 0.25 - 0.07 = 0.30$$

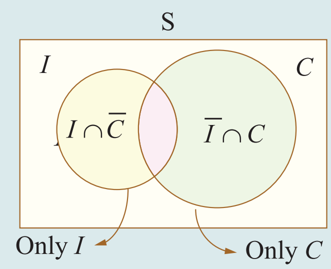
(*ii*) P(only one of the two jobs) $=$ P[only I or only C].

$$= P(I \cap \overline{C}) + P(\overline{I} \cap C)$$

$$= \{P(I) - P(I \cap C)\} + \{P(C) - P(I \cap C)\}$$

$$= \{0.12 - 0.07\} + \{0.25 - 0.07\}$$

$$= 0.23.$$



EXERCISE 12.2

(1) If A and B are mutually exclusive events $P(A) = \frac{3}{8}$ and $P(B) = \frac{1}{8}$, then

find (i) $P(\overline{A})$ (ii) $P(A \cup B)$ (iii) $P(\overline{A} \cap B)$ (iv) $P(\overline{A} \cup \overline{B})$

(2) If A and B are two events associated with a random experiment for which

$P(A) = 0.35$, $P(A \text{ or } B) = 0.85$, and $P(A \text{ and } B) = 0.15$.

Find (i) $P(\text{only } B)$ (ii) $P(\overline{B})$ (iii) $P(\text{only } A)$

(3) A die is thrown twice. Let A be the event, 'First die shows 5' and B be the event, 'second die shows 5'. Find $P(A \cup B)$.

(4) The probability of an event A occurring is 0.5 and B occurring is 0.3. If A and B are mutually exclusive events, then find the probability of

(i) $P(A \cup B)$ (ii) $P(A \cap \overline{B})$ (iii) $P(\overline{A} \cap B)$.

(5) A town has 2 fire engines operating independently. The probability that a fire engine is available when needed is 0.96.

(i) What is the probability that a fire engine is available when needed?

(ii) What is the probability that neither is available when needed?

(6) The probability that a new railway bridge will get an award for its design is 0.48, the probability that it will get an award for the efficient use of materials is 0.36, and that it will get both awards is 0.2. What is the probability, that (i) it will get at least one of the two awards (ii) it will get only one of the awards.

12.6 Conditional Probability

Illustration 12.8

Consider the following example to understand the concept of conditional probability.

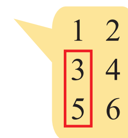
Suppose a fair die is rolled once, then the sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Now we ask two questions

Q_1: What is the probability of getting an odd number which is greater than 2?

Q_2: If the die shows an odd number, then what is the probability that it is greater than 2?

Case 1

The event of getting an odd number which is greater than 2 is $\{3, 5\}$.

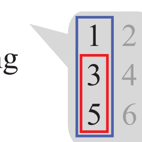


Let P_1 be the probability of getting an odd number which is greater than 2

$$P_1 = \frac{n(\{3,5\})}{n(\{1,\ 2,\ 3,\ 4,\ 5,\ 6\})} = \frac{2}{6} = \frac{1}{3}.$$

Case 2

'If the die shows an odd number' means we restrict our sample space S to a subset containing only odd number.



That is, $S_1 = \{1, 3, 5\}$. Then our interest is to find the probability of the event getting an odd number greater than 2. Let it be P_2

$$P_2 = \frac{n(\{3,5\})}{n(\{\ 1,\ 3,\ 5\})} = \frac{2}{3}$$

In the above two cases the favourable events are the same, but the number of exhaustive outcomes are different. In case 2, we observe that we have first imposed a condition on sample space, then asked to find the probability. This type of probability is called conditional probability.

This can be written by using sample space as

$$P_2 = \frac{\dfrac{n(\{3,5\})}{n(\{1,2,3,4,5,6\})}}{\dfrac{n(\{1,3,5\})}{n(\{1,2,3,4,5,6\})}} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}$$

Important note: Sample space is same for probability and conditional probability.

Definition 12.14

The **conditional probability** of an event B, assuming that the event A has already happened is denoted by $P(B/A)$ and is defined as

$$\boxed{P(B/A) = \frac{P(A\cap B)}{P(A)},\quad \text{provided } P(A) \neq 0}$$

Similarly,

$$\boxed{P(A/B) = \frac{P(A\cap B)}{P(B)},\quad \text{provided } P(B) \neq 0}$$

Example 12.16

If $P(A) = 0.6$, $P(B) = 0.5$, and $P(A\cap B) = 0.2$

Find (i) $P(A/B)$ (ii) $P(\overline{A}/B)$ (iii) $P(A/\overline{B})$.

Solution

Given that $P(A)=0.6$, $P(B)=0.5$, and $P(A\cap B)=0.2$

(i) $$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{0.2}{0.5}=\frac{2}{5}$$

(ii) $$P(\overline{A}/B)=\frac{P(\overline{A}\cap B)}{P(B)}$$

$$=\frac{P(B)-P(A\cap B)}{P(B)}$$

$$=\frac{0.5-0.2}{0.5}=\frac{0.3}{0.5}=\frac{3}{5}.$$

(iii) $$P(A/\overline{B})=\frac{P(A\cap\overline{B})}{P(\overline{B})}$$

$$=\frac{P(A)-P(A\cap B)}{1-P(B)}$$

$$=\frac{0.6-0.2}{1-0.5}=\frac{0.4}{0.5}=\frac{4}{5}.$$

Note 12.5

$$P(A/B)+P(\overline{A}/B)=1$$

Example 12.17

A die is rolled. If it shows an odd number, then find the probability of getting 5.

Solution

Sample space $S=\{1, 2, 3, 4, 5, 6\}$.

Let A be the event of die shows an odd number.

Let B be the event of getting 5.

Then, $A=\{1, 3, 5\}$, $B=\{5\}$, and $A\cap B=\{5\}$.

Therefore, $P(A)=\frac{3}{6}$ and $P(A\cap B)=\frac{1}{6}$

$$P(\text{getting } 5 / \text{die shows an odd number})=P(B/A)$$

$$=\frac{P(A\cap B)}{P(A)}=\frac{\frac{1}{6}}{\frac{3}{6}}$$

$$P(B/A)=\frac{1}{3}.$$

Rewriting the definition of conditional probability, we get the 'Multiplication theorem on probability'.

Theorem 12.7

(Multiplication theorem on probability)

The probability of the simultaneous happening of two events A and B is given by

$$\boxed{\begin{array}{c} P(A \cap B) = P(A/B)P(B) \\ \text{or} \\ P(A \cap B) = P(B/A)P(A) \end{array}}$$

12.6.1 Independent Events

Events are said to be independent if occurrence or non-occurrence of any one of the event does not affect the probability of occurrence or non-occurrence of the other events.

Definition 12.15

Two events A and B are said to be independent if and only if

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Note 12.6

(1) This definition is exactly equivalent to

$$P(A/B) = P(A) \quad \text{if } P(B) > 0$$
$$P(B/A) = P(B) \quad \text{if } P(A) > 0$$

(2) The events $A_1, A_2, A_3, \ldots, A_n$ are mutually independent if
$P(A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n) = P(A_1) \cdot P(A_2) \cdot \cdots \cdot P(A_n)$.

Theorem 12.8

If A and B are independent then

(i) $\bar{A}$ and $\bar{B}$ are independent.

(ii) A and $\bar{B}$ are independent.

(iii) $\bar{A}$ and B are also independent.

Proof

(i) To prove $\bar{A}$ and $\bar{B}$ are independent:
Since A and B are independent

$$P(A \cap B) = P(A) \cdot (PB)$$

To prove $\bar{A}$ and $\bar{B}$ are independent, we have to prove

$$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}).$$

By de Morgan's law

$$P(\bar{A} \cap \bar{B}) = P\left(\overline{A \cup B}\right)$$
$$= 1 - P\left(A \cup B\right)$$

$$= 1-\{P(A)+P(B)-P(A\cap B)\}$$

$$= 1-P(A)-P(B)+P(A)\cdot P(B)$$

$$= (1-P(A))(1-P(B))$$

$$= P(\overline{A})\cdot P(\overline{B})$$

Thus $\overline{A}$ and $\overline{B}$ are independent .

Similarly one can prove (ii) and (iii).

Example 12.18

Two cards are drawn from a pack of 52 cards in succession. Find the probability that both are Jack when the first drawn card is (i) replaced (ii) not replaced

Solution

Let A be the event of drawing a Jack in the first draw,

B be the event of drawing a Jack in the second draw.

Case (i)

Card is replaced

$n(A) = 4$ (Jack)

$n(B) = 4$ (Jack)

and $n(S) = 52$ (Total)

Clearly the event A will not affect the probability of the occurrence of event B and therefore A and B are independent.

$$P(A\cap B) = P(A)\,.\,P(B)$$

$$P(A) = \frac{4}{52}\,,\ P(B)=\frac{4}{52}$$

$$P(A\cap B) = P(A)\ P(B)$$

$$= \frac{4}{52}\cdot\frac{4}{52}$$

$$= \frac{1}{169}\,.$$

Case (ii)

Card is not replaced

In the first draw, there are 4 Jacks and 52 cards in total. Since the Jack, drawn at the first draw is not replaced, in the second draw there are only 3 Jacks and 51 cards in total. Therefore the first event A affects the probability of the occurrence of the second event B.

Thus A and B are not independent. That is, they are dependent events.

Therefore, $P(A\cap B) = P(A)\,.\,P(B/A)$

$$P(A) = \frac{4}{52}$$

$$P(B/A) = \frac{3}{51}$$

$$P(A \cap B) = P(A)\,.\,P(B/A)$$

$$= \frac{4}{52}\cdot\frac{3}{51}$$

$$= \frac{1}{221}\,.$$

Example 12.19

A coin is tossed twice. Events E and F are defined as follows

E= Head on first toss, F = Head on second toss. Find

(i) $P(E \cup F)$ (ii) $P(E / F)$

(iii) $P(\overline{E} / F)$. (iv) Are the events E and F independent?

Solution

The sample space is $S = \{H, T\} \times \{H, T\}$

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

and $E = \{(H, H), (H, T)\}$

$$F = \{(H, H), (T, H)\}$$

$$E \cup F = \{(H, H), (H, T), (T, H)\}$$

$$E \cap F = \{(H, H)\}$$

(i) $P(E \cup F) = P(E) + P(F) - P(E \cap F)$ or $\left(= \dfrac{n(E \cup F)}{n(S)}\right)$

$$= \frac{2}{4} + \frac{2}{4} - \frac{1}{4} = \frac{3}{4}\,.$$

(ii) $P(E / F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{(1/4)}{(2/4)} = \dfrac{1}{2}$

(iii) $P(\overline{E} / F) = \dfrac{P(\overline{E} \cap F)}{P(F)}$

$$= \frac{P(F) - P(E \cap F)}{P(F)}$$

$$= \frac{(2/4) - (1/4)}{(2/4)}$$

$$= \frac{1}{2}$$

(iv) Are the events E and F independent?

We have $P(E \cap F) = \frac{1}{4}$

$$P(E) = \frac{2}{4}, \quad P(F) = \frac{2}{4}$$

$$P(E)\,P(F) = \frac{2}{4}.\frac{2}{4} = \frac{1}{4}$$

$$\Rightarrow P(E \cap F) = P(E)\,.\,P(F)$$

Therefore E and F are independent events.

Note 12.7

Independent events is a property of probability but mutual exclusiveness is a set-theoretic property. Therefore independent events can be identified by their probabilities and mutually exclusive events can be identified by their events.

Theorem 12.9

Suppose A and B are two events, such that $P(A) \neq 0$, $P(B) \neq 0$.

(1) If A and B are mutually exclusive, they cannot be independent.

(2) If A and B are independent they cannot be mutually exclusive. (Without proof)

Example 12.20

If A and B are two independent events such that $P(A) = 0.4$ and $P(A \cup B) = 0.9$. Find $P(B)$.

Solution

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A).P(B) \qquad \text{(since } A \text{ and } B \text{ are independent)}$$

That is, $0.9 = 0.4 + P(B) - (0.4)\,P(B)$

$$0.9 - 0.4 = (1 - 0.4)\,P(B)$$

Therefore, $P(B) = \frac{5}{6}$.

Example 12.21

An anti-aircraft gun can take a maximum of four shots at an enemy plane moving away from it. The probability of hitting the plane in the first, second, third, and fourth shot are respectively 0.2, 0.4, 0.2 and 0.1. Find the probability that the gun hits the plane.

Solution

Let H_1, H_2, H_3 and H_4 be the events of hitting the plane by the anti-aircraft gun in the first second, third and fourth shot respectively.

Let H be the event that anti-aircraft gun hits the plane. Therefore $\overline{H}$ is the event that the plane is not shot down. Given that

$$P(H_1) = 0.2 \qquad \Rightarrow P(\overline{H}_1) = 1 - P(H_1) = 0.8$$

$$P(H_2)=0.4 \quad \Rightarrow P(\overline{H}_2)=1-P(H_2)=0.6$$

$$P(H_3)=0.2 \quad \Rightarrow P(\overline{H}_3)=1-P(H_3)=0.8$$

$$P(H_4)=0.1 \quad \Rightarrow P(\overline{H}_4)=1-P(H_4)=0.9$$

The probability that the gun hits the plane is

$$P(H)=1-P(\overline{H})=1-P(\overline{H_1\cup H_2\cup H_3\cup H_4})$$

$$=1-P\left(\overline{H}_1\cap\overline{H}_2\cap\overline{H}_3\cap\overline{H}_4\right)$$

$$=1-P(\overline{H}_1)P(\overline{H}_2)P(\overline{H}_3)P(\overline{H}_4)$$

$$=1-(0.8)(0.6)(0.8)(0.9)=1-0.3456$$

$$P(H)=0.6544$$

Example 12.22

X speaks truth in 70 percent of cases, and Y in 90 percent of cases. What is the probability that they likely to contradict each other in stating the same fact?

Solution

Let A be the event of X speaks the truth, B be the event of Y speaks the truth

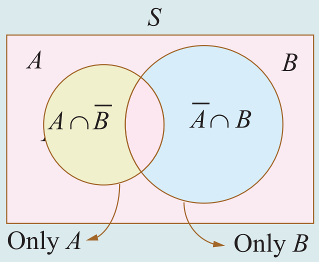
$\therefore\ \overline{A}$ is the event of X not speaking the truth and $\overline{B}$ is the event of Y not speaking the truth.

Let C be the event that they will contradict each other.

Given that

$$P(A)=0.70 \quad \Rightarrow P(\overline{A})=1-P(A)=0.30$$

$$P(B)=0.90 \quad \Rightarrow P(\overline{B})=1-P(B)=0.10$$



$C=$ (A speaks truth and B does not speak truth or B speaks truth and A does not speak truth)

$$C=[(A\cap\overline{B})\cup(\overline{A}\cap B)] \quad \text{(see figure)}$$

since $(A\cap\overline{B})$ and $(\overline{A}\cap B)$ are mutually exclusive,

$$P(C)=P(A\cap\overline{B})+P(\overline{A}\cap B)$$

$$=P(A)\,P(\overline{B})+P(\overline{A})\,P(B)$$

(Since A, B are independent event, A, $\overline{B}$ are also independent events)

$$=(0.70)\ (0.10)+(0.30)\ (0.90)$$

$$=0.070+0.270=0.34$$

$$P(C)=0.34.$$

Example 12.23

A main road in a City has 4 crossroads with traffic lights. Each traffic light opens or closes the traffic with the probability of 0.4 and 0.6 respectively. Determine the probability of

(i) a car crossing the first crossroad without stopping
(ii) a car crossing first two crossroads without stopping
(iii) a car crossing all the crossroads, stopping at third cross.
(iv) a car crossing all the crossroads, stopping at exactly one cross.

Solution:

Let A_i be the event that the traffic light opens at i th cross, for i= 1, 2, 3, 4.

Let B_i be the event that the traffic light closes at i th cross, for i= 1, 2, 3, 4.

The traffic lights are all independent.

Therefore A_i and B_i are all independent events, for i= 1, 2, 3, 4.

Given that

$$P(A_i) = 0.4,\ i = 1, 2, 3, 4$$

$$P(B_i) = 0.6,\ i = 1, 2, 3, 4$$

(i) Probability of car crossing the first crossroad without stopping,

$$P(A_1) = 0.4.$$

(ii) Probability of car crossing first two crossroads without stopping,

$$P(A_1 \cap A_2) = P(A_1 A_2) = (0.4)(0.4) = 0.16$$

(iii) Probability of car crossing all the crossroads, stopping at third cross

$$P(A_1 \cap A_2 \cap B_3 \cap A_4) = P(A_1 A_2 B_3 A_4) = (0.4)(0.4)(0.6)(0.4) = 0.0384$$

(iv) Probability of car crossing all the crossroads, stopping at exactly one of the crossroads is

$$P(B_1A_2A_3A_4 \cup A_1B_2A_3A_4 \cup A_1A_2B_3A_4 \cup A_1A_2A_3B_4)$$

$$= P(B_1A_2A_3A_4) + P(A_1B_2A_3A_4) + P(A_1A_2B_3A_4) + P(A_1A_2A_3B_4)$$

$$= 4(0.4)(0.4)(0.6)(0.4) = 4(0.0384) = 0.1536$$

EXERCISE 12.3

(1) Can two events be mutually exclusive and independent simultaneously?

(2) If A and B are two events such that $P(A \cup B) = 0.7$, $P(A \cap B) = 0.2$, and $P(B) = 0.5$, then show that A and B are independent.

(3) If A and B are two independent events such that $P(A \cup B) = 0.6$, $P(A) = 0.2$, find $P(B)$.

(4) If $P(A) = 0.5$, $P(B) = 0.8$ and $P(B/A) = 0.8$, find $P(A / B)$ and $P(A \cup B)$.

(5) If for two events A and B, $P(A) = \frac{3}{4}$, $P(B) = \frac{2}{5}$ and $A \cup B = S$ (sample space), find the conditional probability $P(A / B)$.

(6) A problem in Mathematics is given to three students whose chances of solving it are $\frac{1}{3}, \frac{1}{4}$, and $\frac{1}{5}$ (i) What is the probability that the problem is solved? (ii) What is the probability that exactly one of them will solve it?

(7) The probability that a car being filled with petrol will also need an oil change is 0.30; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and filter need changing is 0.15.

(i) If the oil had to be changed, what is the probability that a new oil filter is needed?

(ii) If a new oil filter is needed, what is the probability that the oil has to be changed?

(8) One bag contains 5 white and 3 black balls. Another bag contains 4 white and 6 black balls. If one ball is drawn from each bag, find the probability that (i) both are white (ii) both are black (iii) one white and one black

(9) Two thirds of students in a class are boys and rest girls. It is known that the probability of a girl getting a first grade is 0.85 and that of boys is 0.70. Find the probability that a student chosen at random will get first grade marks.

(10) Given $P(A) = 0.4$ and $P(A \cup B) = 0.7$. Find $P(B)$ if

(i) A and B are mutually exclusive (ii) A and B are independent events

(iii) $P(A / B) = 0.4$ (iv) $P(B / A) = 0.5$

(11) A year is selected at random. What is the probability that

(i) it contains 53 Sundays (ii) it is a leap year which contains 53 Sundays

(12) Suppose the chances of hitting a target by a person X is 3 times in 4 shots, by Y is 4 times in 5 shots, and by Z is 2 times in 3 shots. They fire simultaneously exactly one time. What is the probability that the target is damaged by exactly 2 hits?

12.7 Total Probability of an event

Theorem 12.10 (Total Probability of an event)

If $A_1, A_2, A_3, ..., A_n$ are mutually exclusive and exhaustive events and B is any event in S then $P(B)$ is called the total probability of event B and

$$\boxed{P(B) = P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \cdots + P(A_n).P(B/A_n) = \sum_{i=1}^{n} P(A_i).P(B/A_i)}$$

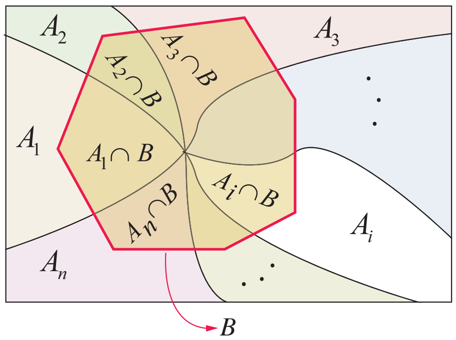
Proof

Since B is any event in S, from the figure shown here

$B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots (A_n \cap B)$.

Since $A_1, A_2, A_3 \ldots A_n$ are mutually exclusive,

$(A_1 \cap B), (A_2 \cap B), (A_3 \cap B), \cdots, (A_n \cap B)$ are also mutually exclusive.



Therefore,

$$P(B) = P[(A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \cdots \cup (A_n \cap B)]$$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + \cdots + P(A_n \cap B)$$

$$P(B) = P(A_1).P(B/A_1) + P(A_2).P(B/A_2) + \cdots + P(A_n).P(B/A_n) = \sum_{i=1}^{n} P(A_i).P(B/A_i).$$

The following problems are solved using the law of total probability of an event.

Example 12.24

Urn-I contains 8 red and 4 blue balls and urn-II contains 5 red and 10 blue balls. One urn is chosen at random and two balls are drawn from it. Find the probability that both balls are red.

Solution

	Red balls	Blue balls	Total
Urn-I	8	4	12
Urn-II	5	10	15
Total	13	14	27

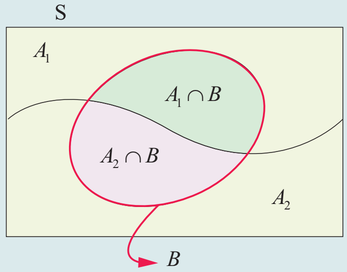
Let A_1 be the event of selecting urn-I and A_2 be the event of selecting urn-II.

Let B be the event of selecting 2 red balls.

We have to find the total probability of event B. That is, $P(B)$.

Clearly A_1 and A_2 are mutually exclusive and exhaustive events.

We have



$$P(A_1) = \frac{1}{2}, \quad P(B/A_1) = \frac{8c_2}{12c_2} = \frac{14}{33}$$

$$P(A_2) = \frac{1}{2}, \quad P(B/A_2) = \frac{5c_2}{15c_2} = \frac{2}{21}$$

We know

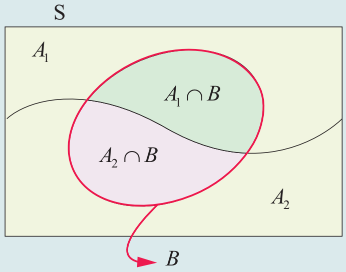
$$P(B) = P(A_1) \,.\, P(B/A_1) + P(A_2) \,.\, P(B/A_2)$$

$$P(B) = \frac{1}{2}.\frac{14}{33} + \frac{1}{2}.\frac{2}{21} = \frac{20}{77}.$$

Example 12.25

A factory has two machines I and II. Machine-I produces 40% of items of the output and Machine-II produces 60% of the items. Further 4% of items produced by Machine-I are defective and 5% produced by Machine-II are defective. If an item is drawn at random, find the probability that it is a defective item.

Solution



Let A_1 be the event that the items are produced by Machine-I, A_2 be the event that items are produced by Machine-II. Let B be the event of drawing a defective item.

We have to find the total probability of event B. That is, $P(B)$.

Clearly A_1 and A_2 are mutually exclusive and exhaustive events.

Therefore, $P(B) = P(A_1) \,.\, P(B/A_1) + P(A_2) \,.\, P(B/A_2)$

We have $\quad P(A_1) = 0.40, \quad P(B/A_1) = 0.04$

$P(A_2) = 0.60, \quad P(B/A_2) = 0.05$

$$P(B) = P(A_1) \,.\, P(B/A_1) + P(A_2) \,.\, P(B/A_2)$$

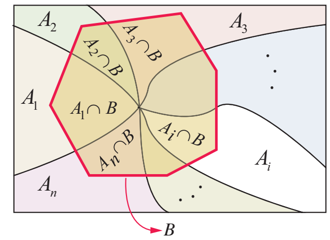
$$= (0.40)(0.04) + (0.60)(0.05)$$

$$= 0.046.$$

12.8 Bayes' Theorem

1702-1761

Thomas Bayes was an English statistician, philosopher and Presbyterian minister who is known for formulating a specific case of a theorem. Bayesian methods stem from the principle of linking prior (before conducting experiment) probability and conditional probability (likelihood) to posterior (after conducting experiment) probability via Bayes' rule. Bayesian probability is the name given to several related interpretations of probability as an amount of epistemic confidence – the strength of beliefs, hypotheses etc., rather than a frequency.



Theorem 12.11 (Bayes' Theorem)

If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive and exhaustive events such that $P(A_i) > 0$, $i = 1,2,3,\ldots.n$ and B is any event in which $P(B) > 0$, then

$$\boxed{P(A_i / B) = \frac{P(A_i)\,P(B / A_i)}{P(A_1)\,P(B / A_1) + P(A_2)\,P(B / A_2) + \cdots + P(A_n)\,P(B / A_n)}}$$

Proof

By the law of total probability of B we have

$$P(B) = P(A_1)\cdot P(B / A_1) + P(A_2)\cdot P(B / A_2) + \cdots + P(A_n)\cdot P(B / A_n)$$

and by multiplication theorem $P(A_i \cap B) = P(B/A_i)P(A_i)$

By the definition of conditional probability,

$$P(A_i / B) = \frac{P(A_i \cap B)}{P(B)}$$

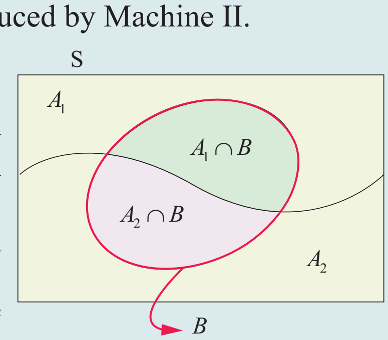
$$P(A_i / B) = \frac{P(B/A_i)P(A_i)}{P(A_1)\cdot P(B / A_1) + P(A_2)\cdot P(B / A_2) + \cdots + P(A_n)\cdot P(B / A_n)} \quad \text{(using formulae)}$$

The above formula gives the relationship between $P(A_i / B)$ and $P(B / A_i)$

Example 12.26

A factory has two machines I and II. Machine I produces 40% of items of the output and Machine II produces 60% of the items. Further 4% of items produced by Machine I are defective and 5% produced by Machine II are defective. An item is drawn at random. If the drawn item is defective, find the probability that it was produced by Machine II. (See the previous example, compare the questions).

Solution



Let A_1 be the event that the items are produced by Machine-I, A_2 be the event that items are produced by Machine-II. Let B be the event of drawing a defective item. Now we are asked to find the conditional probability $P(A_2/B)$. Since A_1, A_2 are mutually exclusive and exhaustive events, by Bayes' theorem,

$$P(A_2 / B) = \frac{P(A_2)\,P(B / A_2)}{P(A_1)\,P(B / A_1) + P(A_2)\,P(B / A_2)}$$

We have,

$$P(A_1) = 0.40, \quad P(B/A_1) = 0.04$$

$$P(A_2) = 0.60, \quad P(B/A_2) = 0.05$$

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

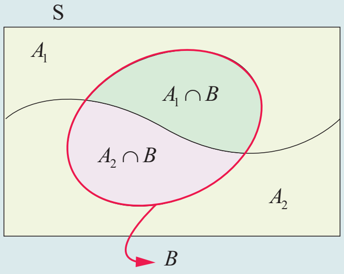
$$P(A_2/B) = \frac{(0.60)(0.05)}{(0.40)(0.04)+(0.60)(0.05)} = \frac{15}{23}.$$

Example 12.27

A construction company employs 2 executive engineers. Engineer-1 does the work for 60% of jobs of the company. Engineer-2 does the work for 40% of jobs of the company. It is known from the past experience that the probability of an error when engineer-1 does the work is 0.03, whereas the probability of an error in the work of engineer-2 is 0.04. Suppose a serious error occurs in the work, which engineer would you guess did the work?

Solution

Let A_1 and A_2 be the events of job done by engineer-1 and engineer-2 of the company respectively. Let B be the event that the error occurs in the work.



We have to find the conditional probability

$P(A_1/B)$ and $P(A_2/B)$ to compare their errors in their work.

From the given information, we have

$$P(A_1) = 0.60, \quad P(B/A_1) = 0.03$$

$$P(A_2) = 0.40, \quad P(B/A_2) = 0.04$$

A_1 and A_2 are mutually exclusive and exhaustive events.

Applying Bayes' theorem,

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$= \frac{(0.60)(0.03)}{(0.60)(0.03)+(0.40)(0.04)}$$

$$P(A_1/B) = \frac{9}{17}.$$

$$P(A_2/B) = \frac{P(A_2)\,P(B/A_2)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2)}$$

$$P(A_2/B) = \frac{(0.40)(0.04)}{(0.60)(0.03)+(0.40)(0.04)}$$

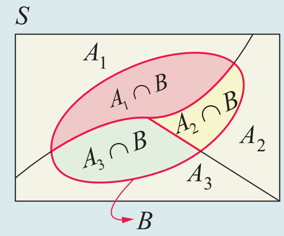
$$P(A_2/B) = \frac{8}{17}.$$

Since $P(A_1/B) > P(A_2/B)$, the chance of error done by engineer-1 is greater than the chance of error done by engineer-2. Therefore one may guess that the serious error would have been be done by engineer-1.

Example 12.28

The chances of X, Y and Z becoming managers of a certain company are 4 : 2 : 3. The probabilities that bonus scheme will be introduced if X, Y and Z become managers are 0.3, 0.5 and 0.4 respectively. If the bonus scheme has been introduced, what is the probability that Z was appointed as the manager?

Solution



Let A_1, A_2 and A_3 be the events of X, Y and Z becoming managers of the company respectively. Let B be the event that the bonus scheme will be introduced.

We have to find the conditional probability $P(A_3 / B)$.

Since A_1, A_2 and A_3 are mutually exclusive and exhaustive events, applying Bayes' theorem

We have
$$P(A_3 / B) = \frac{P(A_3)\, P(B / A_3)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2) + P(A_3)\, P(B / A_3)}$$

$$P(A_1) = \frac{4}{9}, \quad P(B / A_1) = 0.3$$

$$P(A_2) = \frac{2}{9}, \quad P(B / A_2) = 0.5$$

$$P(A_3) = \frac{3}{9}, \quad P(B / A_3) = 0.4$$

$$P(A_3 / B) = \frac{P(A_3)\, P(B / A_3)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2) + P(A_3)\, P(B / A_3)}$$

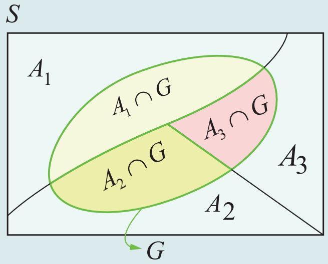
$$P(A_3 / B) = \frac{\left(\frac{3}{9}\right)(0.4)}{\left(\frac{4}{9}\right)(0.3) + \left(\frac{2}{9}\right)(0.5) + \left(\frac{3}{9}\right)(0.4)}$$

$$= \frac{12}{34} = \frac{6}{17}.$$

Example 12.29

A consulting firm rents car from three agencies such that 50% from agency L, 30% from agency M and 20% from agency N. If 90% of the cars from L, 70% of cars from M and 60% of the cars from N are in good conditions (i) what is the probability that the firm will get a car in good condition? (ii) if a car is in good condition, what is probability that it has come from agency N?

Solution



Let A_1, A_2, and A_3 be the events that the cars are rented from the agencies L, M and N respectively.

Let G be the event of getting a car in good condition.

We have to find

(i) the total probability of event G that is, $P(G)$

(ii) find the conditional probability A_3 given G that is, $P(A_3 / G)$

We have

$$P(A_1) = 0.50, \quad P(G/A_1) = 0.90$$

$$P(A_2) = 0.30, \quad P(G/A_2) = 0.70$$

$$P(A_3) = 0.20, \quad P(G/A_3) = 0.60.$$

(i) Since A_1, A_2 and A_3 are mutually exclusive and exhaustive events and G is an event in S, then the total probability of event G is P(G).

$$P(G) = P(A_1)\, P(G/A_1) + P(A_2)\, P(G/A_2) + P(A_3)\, P(G/A_3)$$

$$P(G) = (0.50)(0.90) + (0.30)(0.70) + (0.20)\ (0.60)$$

$$P(G) = 0.78.$$

(ii) The conditional probability A_3 given G is $P(A_3 / G)$

By Bayes' theorem,

$$P(A_3/G) = \frac{P(A_3)\, P(G/A_3)}{P(A_1)\, P(G/A_1) + P(A_2)\, P(G/A_2) + P(A_3)\, P(G/A_3)}$$

$$P(A_3/G) = \frac{(0.20)(0.60)}{(0.50)(0.90) + (0.30)(0.70) + (0.20)\ (0.60)}$$

$$= \frac{2}{13}.$$

EXERCISE 12.4

(1) A factory has two Machines-I and II. Machine-I produces 60% of items and Machine-II produces 40% of the items of the total output. Further 2% of the items produced by Machine-I are defective whereas 4% produced by Machine-II are defective. If an item is drawn at random what is the probability that it is defective?

(2) There are two identical urns containing respectively 6 black and 4 red balls, 2 black and 2 red balls. An urn is chosen at random and a ball is drawn from it. (i) find the probability that the ball is black (ii) if the ball is black, what is the probability that it is from the first urn?

(3) A firm manufactures PVC pipes in three plants viz, *X, Y* and *Z*. The daily production volumes from the three firms *X*, *Y* and *Z* are respectively 2000 units, 3000 units and 5000 units. It is known from the past experience that 3% of the output from plant *X*, 4% from plant *Y* and 2% from plant *Z* are defective. A pipe is selected at random from a day's total production,

(i) find the probability that the selected pipe is a defective one.

(ii) if the selected pipe is a defective, then what is the probability that it was produced by plant *Y* ?

(4) The chances of A, B and C becoming manager of a certain company are 5 : 3 : 2. The probabilities that the office canteen will be improved if A, B, and C become managers are 0.4, 0.5 and 0.3 respectively. If the office canteen has been improved, what is the probability that B was appointed as the manager?

(5) An advertising executive is studying television viewing habits of married men and women during prime time hours. Based on the past viewing records he has determined that during prime time wives are watching television 60% of the time. It has also been determined that when the wife is watching television, 40% of the time the husband is also watching. When the wife is not watching the television, 30% of the time the husband is watching the television. Find the probability that (i) the husband is watching the television during the prime time of television (ii) if the husband is watching the television, the wife is also watching the television.

BF6I7Q

EXERCISE 12.5

Choose the correct or most suitable answer from the given four alternatives

(1) Four persons are selected at random from a group of 3 men, 2 women and 4 children. The probability that exactly two of them are children is

(1) $\frac{3}{4}$ (2) $\frac{10}{23}$ (3) $\frac{1}{2}$ (4) $\frac{10}{21}$

(2) A number is selected from the set $\{1,2,3,...,20\}$. The probability that the selected number is divisible by 3 or 4 is

(1) $\frac{2}{5}$ (2) $\frac{1}{8}$ (3) $\frac{1}{2}$ (4) $\frac{2}{3}$

(3) A, B, and C try to hit a target simultaneously but independently. Their respective probabilities of hitting the target are $\frac{3}{4}, \frac{1}{2}, \frac{5}{8}$. The probability that the target is hit by A or B but not by C is

(1) $\frac{21}{64}$ (2) $\frac{7}{32}$ (3) $\frac{9}{64}$ (4) $\frac{7}{8}$

(4) If A and B are any two events, then the probability that exactly one of them occur is

(1) $P(A\cup\overline{B})+P(\overline{A}\cup B)$ (2) $P(A\cap\overline{B})+P(\overline{A}\cap B)$

(3) $P(A)+P(B)-P(A\cap B)$ (4) $P(A)+P(B)+2P(A\cap B)$

(5) Let A and B be two events such that $P\left(\overline{A\cup B}\right)=\frac{1}{6}$, $P\left(A\cap B\right)=\frac{1}{4}$ and $P\left(\overline{A}\right)=\frac{1}{4}$. Then the events A and B are

(1) Equally likely but not independent (2) Independent but not equally likely

(3) Independent and equally likely (4) Mutually inclusive and dependent

(6) Two items are chosen from a lot containing twelve items of which four are defective, then the probability that at least one of the item is defective

(1) $\frac{19}{33}$ (2) $\frac{17}{33}$ (3) $\frac{23}{33}$ (4) $\frac{13}{33}$

(7) A man has 3 fifty rupee notes, 4 hundred rupees notes and 6 five hundred rupees notes in his pocket. If 2 notes are taken at random, what are the odds in favour of both notes being of hundred rupee denomination?

(1) 1:12 (2) 12:1 (3) 13:1 (4) 1:13

(8) A letter is taken at random from the letters of the word 'ASSISTANT' and another letter is taken at random from the letters of the word 'STATISTICS'. The probability that the selected letters are the same is

(1) $\frac{7}{45}$ (2) $\frac{17}{90}$ (3) $\frac{29}{90}$ (4) $\frac{19}{90}$

(9) A matrix is chosen at random from a set of all matrices of order 2, with elements 0 or 1 only. The probability that the determinant of the matrix chosen is non zero will be

(1) $\frac{3}{16}$ (2) $\frac{3}{8}$ (3) $\frac{1}{4}$ (4) $\frac{5}{8}$

(10) A bag contains 5 white and 3 black balls. Five balls are drawn successively without replacement. The probability that they are alternately of different colours is

(1) $\frac{3}{14}$ (2) $\frac{5}{14}$ (3) $\frac{1}{14}$ (4) $\frac{9}{14}$

(11) If A and B are two events such that $A \subset B$ and $P(B) \neq 0$, then which of the following is correct?

(1) $P(A/B) = \frac{P(A)}{P(B)}$ (2) $P(A/B) < P(A)$

(3) $P(A/B) \geq P(A)$ (4) $P(A/B) > P(B)$

(12) A bag contains 6 green, 2 white, and 7 black balls. If two balls are drawn simultaneously, then the probability that both are different colours is

(1) $\frac{68}{105}$ (2) $\frac{71}{105}$ (3) $\frac{64}{105}$ (4) $\frac{73}{105}$

(13) If X and Y be two events such that $P\left(X/Y\right) = \frac{1}{2}$, $P\left(Y/X\right) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$, then $P(X \cup Y)$ is

(1) $\frac{1}{3}$ (2) $\frac{2}{5}$ (3) $\frac{1}{6}$ (4) $\frac{2}{3}$

(14) An urn contains 5 red and 5 black balls. A ball is drawn at random, its colour is noted and is returned to the urn. Moreover, 2 additional balls of the colour drawn are put in the urn and then a ball is drawn at random. The probability that the second ball drawn is red will be

(1) $\frac{5}{12}$ (2) $\frac{1}{2}$ (3) $\frac{7}{12}$ (4) $\frac{1}{4}$

(15) A number x is chosen at random from the first 100 natural numbers. Let A be the event of numbers which satisfies $\frac{(x-10)(x-50)}{x-30} \geq 0$, then $P(A)$ is

(1) 0.20 (2) 0.51 (3) 0.71 (4) 0.70

(16) If two events A and B are independent such that $P(A)=0.35$ and $P(A\cup B)=0.6$, then $P(B)$ is

(1) $\frac{5}{13}$ (2) $\frac{1}{13}$ (3) $\frac{4}{13}$ (4) $\frac{7}{13}$

(17) If two events A and B are such that $P(\overline{A})=\frac{3}{10}$ and $P(A\cap\overline{B})=\frac{1}{2}$, then $P(A\cap B)$ is

(1) $\frac{1}{2}$ (2) $\frac{1}{3}$ (3) $\frac{1}{4}$ (4) $\frac{1}{5}$

(18) If A and B are two events such that $P(A)=0.4, P\left(B\right)=0.8$ and $P\left(B/A\right)=0.6$, then $P\left(\overline{A}\cap B\right)$ is

(1) 0.96 (2) 0.24 (3) 0.56 (4) 0.66

(19) There are three events A, B and C of which one and only one can happen. If the odds are 7 to 4 against A and 5 to 3 against B, then odds against C is

(1) 23: 65 (2) 65: 23 (3) 23: 88 (4) 88: 23

(20) If a and b are chosen randomly from the set $\{1,2,3,4\}$ with replacement, then the probability of the real roots of the equation $x^2+ax+b=0$ is

(1) $\frac{3}{16}$ (2) $\frac{5}{16}$ (3) $\frac{7}{16}$ (4) $\frac{11}{16}$

(21) It is given that the events A and B are such that $P(A)=\frac{1}{4}$, $P\left(A/B\right)=\frac{1}{2}$ and $P\left(B/A\right)=\frac{2}{3}$. Then $P(B)$ is

(1) $\frac{1}{6}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$ (4) $\frac{1}{2}$

(22) In a certain college 4% of the boys and 1% of the girls are taller than 1.8 meter. Further 60% of the students are girls. If a student is selected at random and is taller than 1.8 meters, then the probability that the student is a girl is

(1) $\frac{2}{11}$ (2) $\frac{3}{11}$ (3) $\frac{5}{11}$ (4) $\frac{7}{11}$

(23) Ten coins are tossed. The probability of getting at least 8 heads is

(1) $\frac{7}{64}$ (2) $\frac{7}{32}$ (3) $\frac{7}{16}$ (4) $\frac{7}{128}$

(24) The probability of two events A and B are 0.3 and 0.6 respectively. The probability that both A and B occur simultaneously is 0.18. The probability that neither A nor B occurs is

(1) 0.1 (2) 0.72 (3) 0.42 (4) 0.28

(25) If m is a number such that $m \le 5$, then the probability that quadratic equation $2x^2+2mx+m+1=0$ has real roots is

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

SUMMARY

Let S be the sample space associated with a random experiment and A be an event.

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to } A}{\text{Exhaustive Number of cases in } S}$$

Axioms of probability

Given a finite sample space S and an event A in S, we define $P(A)$, the probability of A, satisfies the following three axioms.

(1) $P(A) \geq 0$

(2) If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

(3) $P(S) = 1$

The probability of the impossible event is zero. That is $\boxed{P(\emptyset) = 0}$

If A and B are any two events and $\overline{B}$ is the complementary events of B, then

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

If A and B are any two events, then

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

The conditional probability of an event B, assuming that the event A has already happened is denoted by $P(B/A)$ and is defined as

$$\boxed{P(B / A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided } P(A) \neq 0}$$

$$\boxed{P(A / B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided } P(B) \neq 0}$$

The probability of the simultaneous happening of two events A and B is given by

$$\boxed{P(A \cap B) = P(A / B)P(B) \ \text{ or } \ P(A \cap B) = P(B / A)P(A)}$$

Two events A and B are said to be **independent** if and only if

$$\boxed{P(A \cap B) = P(A) \, . \, P(B)}$$

Total Probability

If $A_1, A_2, A_3, \ldots A_n$ are mutually exclusive and exhaustive events and B is any event in S then $P(B)$ is called the total probability of event B and

$$\boxed{P(B) = P(A_1).P(B / A_1) + P(A_2).P(B / A_2) + \cdots + P(A_n).P(B / A_n) = \sum_{i=1}^{n} P(A_i).P(B / A_i)}$$

Bayes' Theorem

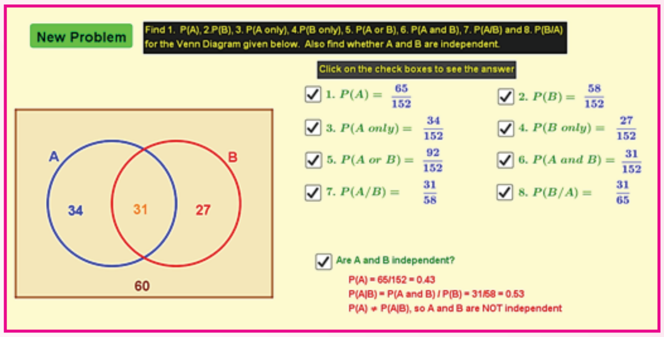
If $A_1, A_2, A_3, \ldots A_n$ are mutually exclusive and exhaustive events such that $P(A_i) > 0$, $i = 1, 2, 3, \ldots . n$ and B is any event in with $P(B) > 0$, then

$$\boxed{P(A_i / B) = \frac{P(A_i)\, P(B / A_i)}{P(A_1)\, P(B / A_1) + P(A_2)\, P(B / A_2) + \ldots + P(A_n)\, P(B / A_n)}}$$

ICT CORNER 12(a)

Probability

Expected Outcome

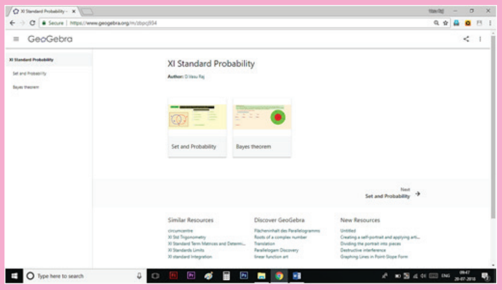


Step 1

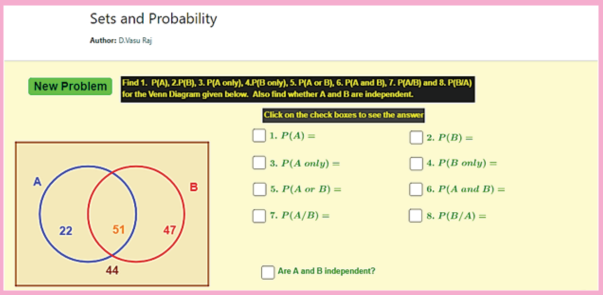
Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Probability" will appear. In that there are several worksheets related to your lesson.

Step 2

Select the work sheet "Sets and Probability". You can change the question by clicking "New Problem". Work out the probabilities and to check your answer, click on the respective check boxes to see the answer.



Step1



Step2

Browse in the link:

XI standard Probability: https://ggbm.at/zbpcj934

B162_11_MAT_EM

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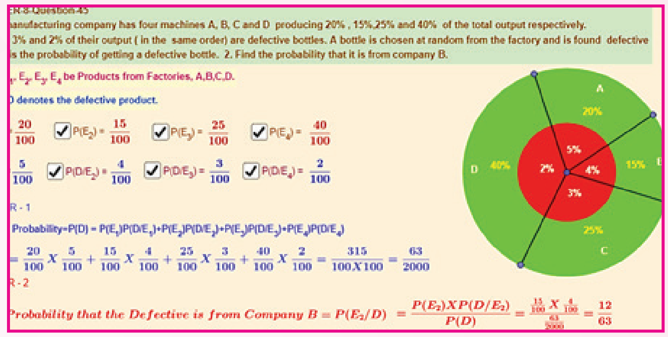
ICT CORNER 12(b)

Introduction to Probability Theory

Expected Outcome



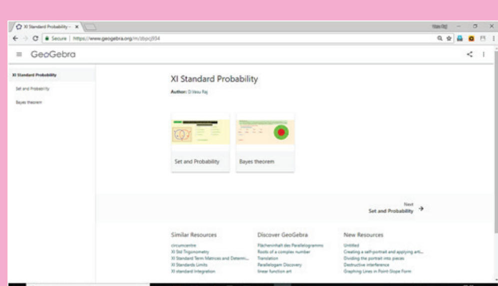
Step 1

Open the Browser type the URL Link given below (or) Scan the QR Code. GeoGebra Workbook called "XI standard Probability" will appear. In that there are several worksheets related to your lesson.

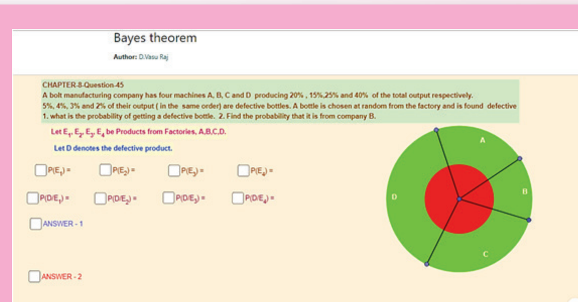
Step 2

Select the work sheet "Bayes Theorem". An example is given. Work out the probabilities step by step as given.

To check your answer, click on the respective check boxes to see the answer.



Step1



Step2

Browse in the link:

XI standard Probability: https://ggbm.at/zbpcj934

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ANSWERS

Exercise 7.1

(1) (i) $\frac{1}{2}\begin{bmatrix} 1 & 9 & 25 \\ 0 & 4 & 16 \end{bmatrix}$ (ii) $\frac{1}{4}\begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 2 & 6 & 10 \\ 5 & 1 & 3 & 7 \end{bmatrix}$ (2) $\pm\sqrt{2}, -3, \frac{1}{2}, 1-\pi$

(3) 5 (4) $A = \frac{1}{3}\begin{bmatrix} -15 & 10 & -8 \\ 10 & -5 & 5 \end{bmatrix}$, $B = \frac{1}{3}\begin{bmatrix} -12 & 2 & -16 \\ 8 & -4 & 13 \end{bmatrix}$

(5) $A^4 = \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$ (6) (ii) $\alpha = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$ (7) $x = 1$

(9) $k = 2$ (12) $-I$ (14) $A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$ (16) 3×4

(18) $A = \begin{bmatrix} 1 & 3 \\ 2 & 12 \\ -5 & 0 \end{bmatrix}$ (19) $x = -2,\ y = -1$ (20) (i) $x = 3^{\frac{1}{3}}$ (ii) $p = -2, q = 0, r = -3$

(21) $A = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$, skew-symmetric (24) Pack I - ₹ 180, Pack II - ₹ 340, Pack III - ₹ 480

Exercise 7.2

(10) 0 (13) 0 (15) (i) 0 (ii) 0 (16) 4
(17) -81 (18) 0 (19) $-1, 2$ (21) 7

Exercise 7.3

(3) $x = 0$ (multiplicity 2), $x = -(a+b+c)$ (5) $x = 0$ (multiplicity 2), $x = -12$

Exercise 7.4

(1) 2.5 sq.units (2) $k = -1, 7$ (3) (i) singular (ii) non-singular (iii) singular

(4) (i) $a = -\frac{6}{7}$ (ii) $b = \frac{49}{8}$ (5) $\frac{1}{2}$ (6) 6

Exercise 7.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	1	1	2	2	2	4	4	2	4	2	4	3	2	4
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
3	3	4	1	3	2	3	3	1	2					

Exercise 8.1

(7) Other sides $\vec{b} - \vec{a}, -\vec{a}, \vec{a} - \vec{b}$ and other diagonal $\vec{b} - 2\vec{a}$

Exercise 8.2

(1) (i) Not direction cosines (ii) direction cosines (iii) Not direction cosines

(2) (i) $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ (ii) $\left(\frac{3}{\sqrt{19}}, \frac{-1}{\sqrt{19}}, \frac{3}{\sqrt{19}}\right)$ (iii) (0, 0, 1)

(3) (i) $\left(\frac{3}{\sqrt{89}}, \frac{-4}{\sqrt{89}}, \frac{8}{\sqrt{89}}\right)$, $(3,-4,8)$ (ii) $\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right)$, $(3,1,1)$

(iii) (0, 1, 0) and (0, 1, 0) (iv) $\left(\frac{5}{\sqrt{2338}}, \frac{-3}{\sqrt{2338}}, \frac{-48}{\sqrt{2338}}\right)$ and $(5,-3-48)$

(v) $\left(\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{-3}{\sqrt{34}}\right)$ and $(3,4,-3)$ (vi) $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$ and $(1,0,-1)$

(4) $\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

(5) $a = \pm\frac{1}{2}$ (6) $a=-1, b=2, c=-1$, or $a=1, b=-2, c=1$ (8) $\lambda = \frac{2}{3}$

(11) (i) $\sqrt{41}$, $\left(\frac{2}{\sqrt{41}}, \frac{1}{\sqrt{41}}, \frac{-6}{\sqrt{41}}\right)$ (ii) $\sqrt{1123}$, $\left(\frac{-15}{\sqrt{1123}}, \frac{27}{\sqrt{1123}}, \frac{13}{\sqrt{1123}}\right)$

(12) $\sqrt{44}+\sqrt{218}+\sqrt{110}$ (13) $\frac{1}{\sqrt{398}}(17\hat{i}-3\hat{j}-10\hat{k})$

(14) yes (16) $m = \pm\frac{1}{\sqrt{3}}$

Exercise 8.3

(1) (i) 9 (ii) 4 (2) (i) $\lambda = \frac{5}{2}$ (ii) $\lambda = -2$ (3) $\theta = \frac{\pi}{4}$

(4) (i) $\theta = \cos^{-1}\left(\frac{-9}{49}\right)$ (ii) $\theta = \frac{2\pi}{3}$ (5) $\theta = \frac{2\pi}{3}$ (8) -55

(11) $5\sqrt{2}$ (12) $\frac{41}{7}$ (13) 5 (14) -42

Exercise 8.4

(1) $\sqrt{507}$ (3) $\frac{\pm 10\sqrt{3}}{\sqrt{35}}(5\hat{i}-3\hat{j}+\hat{k})$ (4) $\pm\frac{(-\hat{i}+2\hat{j}-\hat{k})}{\sqrt{6}}$

(5) $8\sqrt{3}$ sq. units (6) $\frac{1}{2}\sqrt{165}$ sq. units (10) $\frac{\pi}{3}$

Exercise 8.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
3	3	4	2	2	3	4	4	2	3	2	1	1	1	3
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
4	1	4	4	1	3	2	4	3	2					

Exercise 9.1

(1) $\simeq 0.\overline{3}$ (2) $\simeq 0.25$ (3) $\frac{1}{2\sqrt{3}} \simeq 0.288$ (4) $\simeq 0.25$

(5) $\simeq 1$ (6) $\simeq 0$ (7) 1 (8) 3

(9) 2 (10) 3 (11) does not exist (12) does not exist

(13) 0 (14) 1 (15) does not exist

(16) except at $x_0 = 4$ (17) except at $x_0 = \pi$ (19) $f(8^-) = f(8^+) = 25$

(20) No (21) $f(2)$ cannot be concluded (22) 6, 6 (23) does not exist

Exercise 9.2

(1) 32 (2) $\frac{m}{n}$ (3) 108 (4) $\frac{1}{2\sqrt{x}}$

(5) $\frac{1}{6}$ (6) $-\frac{1}{4}$ (7) 3 (8) 4

(9) $\frac{1}{2}$ (10) $-\frac{1}{4}$ (11) $-\frac{3}{4}\sqrt[3]{4}$ (12) 0

(13) $f(x) \to -\infty$ as $x \to 0$ (limit does not exist) (14) $\frac{1}{4}$ (15) $\frac{1}{4a\sqrt{a-b}}$

Exercise 9.3

(1) (i) $f(-2) \to \infty$ as $x \to -2^-$, $f(-2) \to -\infty$ as $x \to -2^+$

(ii) $f\left(\frac{\pi}{2}\right) \to \infty$ as $x \to \frac{\pi}{2}^-$, $f\left(\frac{\pi}{2}\right) \to -\infty$ as $x \to \frac{\pi}{2}^+$

(2) $f(3) \to -\infty$ as $x \to 3^-$, $f(3) \to \infty$ as $x \to 3^+$ (3) $f(x) \to 0$ as $x \to \infty$

(4) 0 (5) $f(x) \to \infty$ as $x \to \infty$ (6) -1 (7) $\frac{1}{4}$

(9) $\frac{1}{\alpha}$ (10) 30

Exercise 9.4

(1) e^7 (2) $e^{\frac{1}{3}}$ (3) 1 (4) $\frac{1}{e^8}$ (5) e^3

(6) $\frac{1}{8}$ (7) $\frac{\alpha}{\beta}$ (8) $\frac{2}{5}$ (9) $\begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m > n \\ f(\alpha) \to \infty \text{ as } \alpha \to 0 & \text{if } m < n \end{cases}$

(10) $2\cos a$ (11) $\frac{b}{a}$ (12) $\frac{2}{3}$ (13) $\frac{1}{2}$ (14) 2

(15) $\log\frac{2}{3}$ (16) $\log 9$ (17) $\frac{1}{2}$ (18) $\log 3 - 1$ (19) a

(20) $-\frac{3}{2}$ (21) e^2 (22) $\frac{1}{4\sqrt{2}}$ (23) 1 (24) e^2

(25) 2 (26) $a - b$ (27) $\frac{1}{2}$ (28) $\frac{1}{2}$

Exercise 9.5

(2) (i) continuous for all $x \in \mathbb{R}$ (ii) continuous in $\mathbb{R}$

(iii) continuous for all $x \in \mathbb{R} - (2n+1)\frac{\pi}{2}, n \in z$ (iv) continuous for all $x \in \mathbb{R}$

(v) continuous for $(0, \infty)$ (vi) continuous for all $x \in \mathbb{R} - \{0\}$

(vii) continuous for all $x \in \mathbb{R} - \{-4\}$ (viii) continuous for all $x \in \mathbb{R}$

(ix) continuous for all $x \in \mathbb{R} - \{-1\}$ (x) continuous for all $x \in \mathbb{R} - \frac{n\pi}{2}, n \in z$

(3) (i) not continuous at $x = 3$ (ii) continuous for all $x \in \mathbb{R}$

(iii) continuous for all $x \in \mathbb{R}$ (iv) continuous for all $x \in \left[0, \frac{\pi}{2}\right)$

(4) (i) continuous at $x_0 = 1$ (ii) not continuous at $x_0 = 3$

(6) $\alpha = 4$ (8) 6

(9) (i) not continuous at $x = 1$ (ii) not continuous at $x = 0$

(10) continuous at $x = 0, 1, 3$

(11) (i) removable discontinuity at $x = -2$, $g(x) = \begin{cases} \frac{x^2 - 2x - 8}{x+2} & \text{if } x \neq -2 \\ -6 & \text{if } x = -2 \end{cases}$

(ii) removable discontinuity at $x = -4$, $g(x) = \begin{cases} \frac{x^3 + 64}{x+4} & \text{if } x \neq -4 \\ 48 & \text{if } x = -4 \end{cases}$

(iii) removable discontinuity at $x = 9$, $g(x) = \begin{cases} \frac{3 - \sqrt{x}}{9 - x} & \text{if } x \neq 9 \\ \frac{1}{6} & \text{if } x = 9 \end{cases}$

(12) -2 (13) $f(0) = 0$ (14) $f(1) = \frac{2}{3}$

Exercise 9.6

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	3	4	1	1	4	2	2	2	3	4	3	4	3	1
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	1	1	4	2	2	2	2	2	4					

Exercise 10.1

(1) (i) 0 (ii) -4 (iii) $-2x$

(2) (i) $f'(1^-) = -1, f'(1^+) = 1$, not differentiable (ii) $f'(x) \to -\infty$ as $x \to 1^-$, not differentiable

(iii) $f'(1^-) = 1, f'(1^+) = 2$, not differentiable

(3) (i) differentiable (ii) not differentiable

(iii) not differentiable (iv) not differentiable

(5) at $x = -1$ and $x = 8$ are cusps

at $x=4$ it is not continuous, at $x=11$, tangent is perpendicular

(6) does not exist

(7) (i) not differentiable at $x=n\pi, n\in z$ (ii) not differentiable at $x=(2n+1)\frac{\pi}{2}, n\in z$

Exercise 10.2

(1) $1-3\cos x$ (2) $\cos x-\sin x$ (3) $x\cos x+\sin x$

(4) $-\sin x-2\sec^2 x$ (5) $3t^2\cos t-t^3\sin t$ (6) $4\sec t\tan t+\sec^2 t$

(7) $e^x(\cos x+\sin x)$ (8) $\dfrac{x\sec^2 x-\tan x}{x^2}$ (9) $\dfrac{1}{1+\cos x}$

(10) $\dfrac{(1-x)\cos x+(1+x)\sin x}{(\sin x+\cos x)^2}$ (11) $\cos x+\sin x$ (12) $\dfrac{x\cos x-2\sin x}{x^3}$

(13) $\tan\theta\sec\theta+\cos\theta+\sin\theta$ (14) $-\dfrac{(1+\cos^2 x)}{\sin^3 x}$ (15) $x\cos 2x+\sin x\cos x$

(16) $e^{-x}\left[\dfrac{1}{x}-\log x\right]$ (17) $e^{-3x}\left[-3(x^2+5)\log(1+x)+\dfrac{x^2+5}{1+x}+2x\log(1+x)\right]$

(18) $\dfrac{\pi}{180}\cos\dfrac{\pi}{180}x$ (19) $\dfrac{\log_{10}e}{x}$

Exercise 10.3

(1) $5(2x+4)(x^2+4x+6)^4$ (2) $3\sec^2 3x$ (3) $-\sec^2 x\sin(\tan x)$

(4) $x^2(1+x^3)^{-\frac{2}{3}}$ (5) $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$ (6) $e^x\cos(e^x)$

(7) $7(3x^2+4)(x^3+4x)^6$ (8) $\dfrac{3}{2}\left(t-\dfrac{1}{t}\right)^{\frac{1}{2}}\left(1+\dfrac{1}{t^2}\right)$ (9) $\dfrac{1}{3}\sec^2 t(1+\tan t)^{-\frac{2}{3}}$

(10) $-3x^2\sin(a^3+x^3)$ (11) $-my$ (12) $20\sec 5x\tan 5x$

(13) $\dfrac{8(2x-5)^3}{(8x^2-5)^4}[-4x^2+30x-5]$ (14) $\dfrac{8x^3+14x}{3(x^2+2)^{\frac{2}{3}}}$ (15) $e^{-x^2}[1-2x^2]$

(16) $-\dfrac{3t^2}{2(t^3+1)^{\frac{3}{4}}(t^3-1)^{\frac{5}{4}}}$ (17) $\dfrac{14-3x}{2(7-3x)\sqrt{7-3x}}$ (18) $-\sin x\sec^2(\cos x)$

(19) $\sin x(1+\sec^2 x)$ (20) $\dfrac{5^{-\frac{1}{x}}(\log 5)}{x^2}$ (21) $\dfrac{\sec^2 x}{\sqrt{1+2\tan x}}$

(22) $3\sin x\cos x(\sin x-\cos x)$ (23) $-k\sin kx\sin(2\cos kx)$ (24) $-6\sin 2x(1+\cos^2 x)^5$

(25) $\dfrac{3e^{3x}+2e^{4x}}{(1+e^x)^2}$ (26) $\dfrac{2\sqrt{x}+1}{4\sqrt{x}\sqrt{x+\sqrt{x}}}$ (27) $e^{x\cos x}[\cos x-x\sin x]$

(28) $\dfrac{4\sqrt{x}\sqrt{x+\sqrt{x}}+2\sqrt{x}+1}{8\sqrt{x}\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$ (29) $\dfrac{\cos\left(\tan\sqrt{\sin x}\right)\sec^2\left(\sqrt{\sin x}\right)\cos x}{2\sqrt{\sin x}}$ (30) $\dfrac{-2}{1+x^2}$

Exercise 10.4

(1) $x^{\cos x}\left(\dfrac{\cos x}{x}-\sin x\log x\right)$ (2) $x^{\log x}\left(\dfrac{2\log x}{x}\right)+(\log x)^x\left[\dfrac{1}{\log x}+\log(\log x)\right]$

(3) $\dfrac{y(2x-1)}{x(1+2y)}$ (4) $\dfrac{y(x\log y-y)}{x(y\log x-x)}$ (5) $(\cos x)^{\log x}\left[\dfrac{\log(\cos x)}{x}-\tan x\log x\right]$

(6) $-\dfrac{b^2x}{a^2y}$ (7) $\dfrac{x\sqrt{x^2+y^2}+y}{x-y\sqrt{x^2+y^2}}$ (8) $\dfrac{1-\sec^2(x+y)-\sec^2(x-y)}{\sec^2(x+y)-\sec^2(x-y)}$

(10) $\dfrac{1}{2}$ (11) $\dfrac{6}{1+9x^2}$ (12) 1 (13) $-\tan t$

(14) $\tan t$ (15) $\dfrac{t^2-1}{2t}$ (16) $\dfrac{2}{1+x^2}$ (17) $\dfrac{3}{\sqrt{1-x^2}}$

(18) 1 (19) $\cos x^2$ (20) 2 (21) $\dfrac{1}{2}$

(22) -1 (23) $\dfrac{x}{(1-x^2)^{\frac{3}{2}}}$

Exercise 10.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
2	4	3	3	1	4	3	2	1	4	3	3	2	2	4
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
2	3	1	4	2	4	1	1	3	2					

Exercise 11.1

(1) (i) $\dfrac{x^{12}}{12}+c$ (ii) $-\dfrac{1}{6x^6}+c$ (iii) $\dfrac{3}{7}x^{\frac{7}{3}}+c$ (iv) $\dfrac{8}{13}x^{\frac{13}{8}}+c$

(2) (i) $-\cot x+c$ (ii) $\sec x+c$ (iii) $-\text{cosec}x+c$ (iv) $\tan x+c$

(3) (i) 12^3x+c (ii) $\log|x|+c$ (iii) e^x+c

(4) (i) $\tan^{-1}x+c$ (ii) $\sin^{-1}x+c$

Exercise 11.2

(1) (i) $\dfrac{(x+5)^7}{7}+c$ (ii) $\dfrac{1}{9(2-3x)^3}+c$ (iii) $\dfrac{2}{9}(3x+2)^{\frac{3}{2}}+c$

(2) (i) $\dfrac{-\cos 3x}{3}+c$ (ii) $-\dfrac{\sin(5-11x)+c}{11}$ (iii) $-\dfrac{\cot(5x-7)}{5}+c$

(3) (i) $\dfrac{1}{3}e^{3x-6}+c$ (ii) $-\dfrac{e^{8-7x}}{7}+c$ (iii) $-\dfrac{1}{4}\log|6-4x|+c$

(4) (i) $5\tan\frac{x}{5}+c$ (ii) $-\frac{1}{5}\text{cosec}(5x+3)+c$ (iii) $-\frac{1}{15}\sec(2-15x)+c$

(5) (i) $\frac{1}{4}\sin^{-1}(4x)+c$ (ii) $\frac{1}{9}\sin^{-1}(9x)+c$ (iii) $\frac{1}{6}\tan^{-1}(6x)+c$

Exercise 11.3

(1) $\frac{(x+4)^6}{6}+\frac{1}{3(2-5x)^3}+\frac{\cot(3x-1)}{3}+c$ (2) $-2\sin(5-2x)+3e^{3x-6}-6\log|6-4x|+c$

(3) $5\tan\frac{x}{5}+9\sin 2x+2\sec(5x+3)+c$ (4) $2\sin^{-1}(4x)+9\sin^{-1}(3x)-3\tan^{-1}(5x)+c$

(5) $2\tan^{-1}(3x+2)+3\sin^{-1}(3-4x)+c$ (6) $\sin\left(\frac{x}{3}-4\right)+\log|7x+9|+5e^{\frac{x}{5}+3}+c$

Exercise 11.4

(1) $2x^2-5x+3$ (2) $3(x^3-x^2-1)$ (3) $2x^3-3x^2+5x+26$

(4) (i) 8 seconds (ii) 39.2 m/sec (iii) 78.4 m/sec

(5) (i) 2.4 sq.cm (ii) 0.4 sq.cm

Exercise 11.5

(1) $\frac{x^2}{2}+4x-3\log|x|-\frac{2}{x}+c$ (2) $\frac{x^2}{2}+\log|x|+2x+c$ (3) $\frac{8x^3}{3}+26x^2-180x+c$

(4) $\tan x-\cot x-2x+c$ (5) $2[\sin x+x\cos\alpha]+c$ (6) $-2\text{cosec}2x+c$

(7) $-3\cot x-4\text{cosec}\,x+c$ (8) $x-\sin x+c$ (9) $2\left[\frac{\sin 3x}{3}+\sin x\right]+c$

(10) $\frac{1}{2}\left[\frac{\sin 5x}{5}+\sin x\right]+c$ (11) $\frac{1}{2}\left[x-\frac{\sin 10x}{10}\right]+c$ (12) $-\frac{1}{8}\cos 4x+c$

(13) $\frac{(ae)^x}{\log(ae)}+c$ (14) $\frac{2}{15}(3x+7)^{\frac{5}{2}}-\frac{2}{3}(3x+7)^{\frac{3}{2}}+c$

(15) $\frac{2^{2x+2}}{\log 2}-\frac{2^{2-3x}}{3\log 2}+c$ (16) $\frac{2}{21}[(x+3)^{\frac{3}{2}}+(x-4)^{\frac{3}{2}}]+c$

(17) $2\log|x+3|-\log|x+2|+c$ (18) $\frac{1}{9}\log|x-1|-\frac{1}{9}\log|x+2|+\frac{1}{3(x+2)}+c$

(19) $\log\left|\frac{x+2}{x-1}\right|+3\tan^{-1}x+c$ (20) $\frac{x^2}{2}+3x-\log|x-1|+8\log|x-2|+c$

Exercise 11.6

(1) $\sqrt{1+x^2}+c$ (2) $\frac{1}{3}\tan^{-1}(x^3)+c$ (3) $\log|e^x+e^{-x}|+c$

(4) $\log|10^x+x^{10}|+c$ (5) $-2\cos\sqrt{x}+c$ (6) $\log|\log(\sin x)|+c$

(7) $\log\left|\log\left(\tan\frac{x}{2}\right)\right|+c$ (8) $\frac{1}{b^2}\log|a^2+b^2\sin^2 x|+c$ (9) $\frac{(\sin^{-1}x)^2}{2}+c$

(10) $(1+\sqrt{x})^2-4(1+\sqrt{x})+2\log\left|1+\sqrt{x}\right|+c$ (11) $\log|\log(\log x)|+c$

(12) $-e^{-\beta x^{\alpha}}+c$ (13) $2\sqrt{\sec x}+c$ (14) $\frac{(1-x)^{19}}{19}-\frac{(1-x)^{18}}{18}+c$

(15) $\frac{\sin^6 x}{6}-\frac{\sin^8 x}{8}+c$ (16) $(x-a)\cos a-\sin a\log|\sec(x-a)|+c$

Exercise 11.7

(1) (i) $e^{3x}[3x-1]+c$ (ii) $-\frac{x\cos 3x}{3}+\frac{\sin 3x}{9}+c$

(iii) $-e^{-5x}[5x+1]+c$ (iv) $x\sec x-\log|\sec x+\tan x|+c$

(2) (i) $\frac{x^2\log|x|}{2}-\frac{x^2}{4}+c$ (ii) $e^{3x}[9x^2-6x+2]+c$

(iii) $x^2\sin x+2x\cos x-2\sin x+c$ (iv) $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

(3) (i) $-\sin^{-1}x\sqrt{1-x^2}+x+c$ (ii) $\frac{1}{2}e^{x^2}[x^4-2x^2+2]+c$

(iii) $\frac{1}{2}\left[4x\tan^{-1}4x-\log\left|\sqrt{1+16x^2}\right|\right]+c$ (iv) $2\left[x\tan^{-1}x-\log\left|\sqrt{1+x^2}\right|\right]+c$

Exercise 11.8

(1) (i) $\frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]+c$ (ii) $\frac{e^{2x}}{5}[2\sin x-\cos x]+c$

(iii) $\frac{e^{-x}}{5}[2\sin 2x-\cos 2x]+c$

(2) (i) $-\frac{e^{-3x}}{13}[3\sin 2x+2\cos 2x]+c$ (ii) $-\frac{e^{-4x}}{10}[2\sin 2x+\cos 2x]+c$

(iii) $\frac{e^{-3x}}{10}[\sin x-3\cos x]+c$

Exercise 11.9

(1) $e^x\log|\sec x|+c$ (2) $\frac{e^x}{2x}+c$ (3) $e^x\sec x+c$

(4) $e^x\tan x+c$ (5) $xe^{\tan^{-1}x}+c$ (6) $\frac{x}{1+\log|x|}+c$

Exercise 11.10

(1) (i) $\frac{1}{4}\log\left|\frac{2+x}{2-x}\right|+c$ (ii) $\frac{1}{20}\log\left|\frac{5+2x}{5-2x}\right|+c$ (iii) $\frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right|+c$

(2) (i) $\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}-3+x}{\sqrt{2}+3-x}\right|+c$ (ii) $\frac{1}{10}\log\left|\frac{x-4}{x+6}\right|+c$ (iii) $\log\left|x+2+\sqrt{x^2+4x+2}\right|+c$

(3) (i) $\log\left|x+2+\sqrt{(x+2)^2-1}\right|+c$ (ii) $\log\left|x-2+\sqrt{x^2-4x+5}\right|+c$ (iii) $\sin^{-1}\left(\frac{x-4}{5}\right)+c$

Exercise 11.11

(1) (i) $\log\ |x^2+4x-12|-\frac{7}{8}\log\left|\frac{x-2}{x+6}\right|+c$ (ii) $\frac{5}{2}\log|x^2+2x+2|-7\tan^{-1}(x+1)+c$

(iii) $\frac{3}{4}\log|2x^2-2x+3|+\frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c$

(2) (i) $5\sin^{-1}\frac{x-2}{\sqrt{13}}-2\sqrt{9+4x-x^2}+c$ (ii) $\sqrt{x^2-1}+2\log\left|x+\sqrt{x^2-1}\right|+c$

(iii) $2\sqrt{x^2+4x+1}-\log\left|x+2+\sqrt{x^2+4x+1}\right|+c$

Exercise 11.12

(1) (i) $\frac{x+1}{2}\sqrt{x^2+2x+10}+\frac{9}{2}\log\left|x+1+\sqrt{x^2+2x+10}\right|+c$

(ii) $\frac{x-1}{2}\sqrt{x^2-2x-3}-2\log\left|x-1+\sqrt{x^2-2x-3}\right|+c$

(iii) $\frac{x-5}{2}\sqrt{10x-x^2-24}+\frac{1}{2}\sin^{-1}(x-5)+c$

(2) (i) $\frac{1}{4}\left[(2x+5)\sqrt{9-(2x+5)^2}+9\sin^{-1}\left(\frac{2x+5}{3}\right)\right]+c$

(ii) $\frac{1}{4}\left[(2x+1)\sqrt{81+(2x+1)^2}+81\log\left|2x+1+\sqrt{81+(2x+1)^2}\right|\right]+c$

(iii) $\frac{x+1}{2}\sqrt{(x+1)^2-4}+2\log\left|x+1+\sqrt{(x+1)^2-4}\right|+c$

Exercise 11.13

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1	3	4	1	3	1	3	2	4	3	4	2	4	4	1
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	3	2	4	1	3	4	3	1	4					

Exercise 12.1

(1) (i) permissible (ii) not permissible (iii) not permissible

(2) (i) $\frac{1}{2}$ (ii) 1 (3) (i) $\frac{5}{9}$ (ii) $\frac{4}{9}$ (4) (i) $\frac{1}{7}$ (ii) $\frac{2}{7}$ (5)(i) $\frac{7}{64}$ (ii) $\frac{247}{256}$ (iii) $\frac{37}{256}$

(6) $\frac{37}{100}$ (7) (i) $\frac{7}{33}$ (ii) $\frac{14}{55}$ (8) (i) $\frac{2}{13}$ (ii) $\frac{5}{13}$ (iii) $\frac{2}{13}$

(9) $\frac{627}{728}$ (10) (i) $\frac{5}{12}$ (ii) 2 to 3

Exercise 12.2

(1) (i) $\frac{5}{8}$ (ii) $\frac{1}{2}$ (iii)) $\frac{1}{8}$ (iv) 1 (2) (i) 0.50 (ii) 0.35 (iii) 0.20

(3) $\frac{11}{36}$ (4) (i) 0.8 (ii) 0.5 (iii) 0.3

(5) (i) 0.9984 (ii) 0.0016 (6) (i) 0.64 (ii) 0.44

Exercise 12.3

(1) No (3) 0.5 (4) (i) 0.5 (ii) 0.9

(5) $\frac{3}{8}$ (6) (i) $\frac{3}{5}$ (ii) $\frac{13}{30}$ (7) (i) 0.5 (ii) 0.375

(8) (i) $\frac{1}{4}$ (ii) $\frac{9}{40}$ (iii) $\frac{21}{40}$ (9) 0.75

(10) (i) 0.3 (ii) 0.5 (iii) 0.5 (iv) 0.5 (11) (i) $\frac{5}{28}$ (ii) $\frac{1}{14}$ (12) $\frac{13}{30}$

Exercise 12.4

(1) 0.028(2) (i) $\frac{11}{20}$ (ii) $\frac{6}{11}$ (3) (i) $\frac{7}{250}$ (ii) $\frac{3}{7}$

(4) $\frac{15}{41}$ (5) (i) $\frac{9}{25}$ (ii) $\frac{2}{3}$

Exercise 12.5

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
4	3	1	2	2	1	1	4	2	3	3	1	4	2	3
(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)					
1	4	3	2	3	2	2	4	4	3					

GLOSSARY

CHAPTER 7
MATRICES and DETERMINANTS

matrix	அணி
order	வரிசை
row matrix	நிரை அணி
column matrix	நிரல் அணி
zero matrix	பூஜ்ஜிய அணி
null matrix	வெற்று அணி
square matrix	சதுர அணி
diagonal matrix	மூலைவிட்ட அணி
unit matrix	அலகு அணி
triangular matrix	முக்கோண வடிவ அணி
upper triangular matrix	மேல் முக்கோண வடிவ அணி
lower triangular matrix	கீழ் முக்கோண வடிவ அணி
principal diagonal	முதன்மை மூல விட்டம்
scalar matrix	திசையிலி அணி
conformable	உகந்த
commutative property	பரிமாற்றுப் பண்பு
associative property	சேர்ப்புப் பண்பு
identity property	சமனிப் பண்பு
inverse property	எதிர்மறைப் பண்பு
distributive property	பங்கீட்டுப் பண்பு
symmetric	சமச்சீர்
skew-symmetric	எதிர் சமச்சீர்
determinant	அணிக்கோவை
singular matrix	பூஜ்ஜியக் கோவை அணி
non-singular matrix	பூஜ்ஜியமற்ற கோவை அணி

CHAPTER 8
VECTOR ALGEBRA

vector	வெக்டர்
initial point	தொடக்கப் புள்ளி
terminal point	முடிவுப் புள்ளி
support of the vector	வெக்டரின் தாங்கி
free vector	கட்டிலா வெக்டர்
localised vector	அறுதியிட்ட வெக்டர்
co-initial vectors	ஒரே தொடக்கப்புள்ளி வெக்டர்கள்
co-terminal vectors	ஒரே முடிவுப்புள்ளி வெக்டர்கள்
collinear vectors	ஒரே கோடமை வெக்டர்கள்
parallel vectors	இணை வெக்டர்கள்
coplanar vectors	ஒரு தள வெக்டர்கள்
equal vectors	சம வெக்டர்கள்
zero vector	பூஜ்ஜிய வெக்டர்
unit vector	அலகு வெக்டர்
like vectors	ஒரே திசை வெக்டர்கள்
unlike vectors	எதிர் திசை வெக்டர்கள்
scalar multiplication	திசையிலிப் பெருக்கம்
position vector	நிலை வெக்டர்
section formula	பிரிவு சூத்திரம்
resolution of vector	வெக்டரைக் கூறுகளாகப் பிரித்தல்
direction cosines	திசைக் கொசைன்கள்
direction ratios	திசை விகிதங்கள்
scalar product	திசையிலிப் பெருக்கம்
vector product	வெக்டர் பெருக்கம்

CHAPTER 9
LIMITS AND CONTINUITY

calculus	நுண் கணிதம்
limit	எல்லை
one sided limit	ஒருபுற எல்லை
left hand limit	இடப்புற எல்லை
right hand limit	வலப்புற எல்லை
infinite limit	முடிவிலா எல்லை
limit at infinity	முடிவிலியில் எல்லை
vertical asymptote	செங்குத்துத் தொலைத் தொடுகோடு

horizontal asymptote	கிடைமட்டத் தொலைத் தொடுகோடு
Sandwich theorem	இடையீட்டுத் தேற்றம்
continuity	தொடர்ச்சித் தன்மை
discontinuity	தொடர்ச்சியின்மைத் தன்மை
removable discontinuity	நீக்கக் கூடிய தொடர்ச்சியின்மை
jump discontinuity	துள்ளல் தொடர்ச்சியின்மை

CHAPTER 10
DIFFERENTIAL CALCULUS - DIFFERENTIABILITY AND METHODS OF DIFFERENTIATION

analytic equation	பகுமுறை சமன்பாடு
derivative	வகைக்கெழு
velocity	திசைவேகம்
acceleration	முடுக்கம்
jerk	குலுக்கம்
tangency	தொடுகோட்டுப் பண்பு
difference quotient	வித்தியாசங்களின் விகிதம்
secant line	வெட்டுக்கோடு
tangent line	தொடுகோடு
slope of the curve	வளைவரை சாய்வு
rectilinear motion	நேர்கோட்டியக்கம்
position function	நிலைச்சார்பு
differentiable	வகைமை / வகையிடத்தக்க
differentiation	வகையிடல்
left hand derivative	இடப்புற வகையிடல்
right hand derivative	வலப்புற வகையிடல்
non-differentiability	வகையிடத் தக்கதல்லாத
quotient rule	வகுத்தல் விதி
chain rule	பிணைப்பு விதி
composite functions rule	சார்புகளின் சேர்ப்பு விதி

function of a function rule	சார்பின் சார்பு விதி
intermediate argument	இடைநிலை மாறி
implicit differentiation	உள்ளார்ந்த சார்பு வகையிடல்
explicit differentiation	வெளிப்படைச் சார்பு வகையிடல்
parametric differentiation	துணையலகு சார்பு வகையிடல்
higher order derivative	உயர்வரிசை வகையிடல்

CHAPTER 11
INTEGRAL CALCULUS

integrand	தொகைச் சார்பு
integrator	தொகை மாறி
anti-derivative	எதிர்மறை வகையிடல்
indefinite integral	எல்லை வரையறுக்கப்படாத தொகை

CHAPTER 12 INTRODUCTION TO PROBABILITY THEORY

experiment	சோதனை
deterministic experiment	நிர்ணயிக்கப்பட்ட சோதனை
random experiment	சமவாய்ப்பு சோதனை
event	நிகழ்ச்சி
sure event or certain event	நிச்சய நிகழ்ச்சி
impossible event	இயலா நிகழ்ச்சி
exhaustive event	யாவுமளாவிய நிகழ்ச்சி
sample space	கூறுவெளி
mutually exclusive events	ஒன்றையொன்று விலக்கும் நிகழ்ச்சிகள்
equally likely events	சமவாய்ப்பு நிகழ்ச்சிகள்
conditional probability	சார்புநிலை நிகழ்தகவு
independent events	சார்பிலா நிகழ்ச்சிகள்

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(17) Mathematical Statistics, Saxena and Kapoor

Mathematics – Volume 2

Text Book Development Team - HSc - Class 11

Domain Experts

Dr. S. Ponnusamy
Professor,
Indian Institute of Technology, Chennai

Dr. K. Srinivasan,
Associate Professor & Head (Rtd),
Department of Mathematics,
Presidency College (Autonomous),
Chennai

Dr. E. Chandrasekaran,
Professor in Mathematics,
Vel Tech Rangarajan Dr. Sagunthala
R&D Institute of Science and Technology
(Deemed to be University), Avadi, Chennai

Dr. C.Selvaraj
Associate Professor of Mathematics
Presidency college, Chennai

Dr. Felbin C. Kennedy
Associate Professor & Head,
Department of Mathematics
Stella Maris College, Chennai

Dr. G. Palani,
Assistant Professor,
Dr. Ambedkar Govt. Arts College,
Vyasarpadi, Chennai.

Content Experts

Dr. R. Vembu,
Associate Professor,
SBK College, Aruppukkottai

Dr. D. J. Prabhakaran
Assistant Professor,
Dept of Mathematics
MIT Campus AnnaUniversity, Chennai.

Reviewers

Dr. V. Thangaraj
Former Director of RIASM
University of Madras

Dr. M. Chandrasekar
Professor and Head
Department of Mathematics
Anna University

Dr. K. C. Sivakumar
Professor Department of Mathematics
IIT Madras, Chennai

Dr. L. Jones Tarcius Doss
Professor Department of Mathematics
Anna University, Chennai

Dr. J. R. V. Edward
Principal Incharge
Department of Mathematics
SCOTT Christian College,
Nagercoil, Kaniyakumari Dt

Dr. R. Roopkumar
Associate Professor
Dept. of Mathematics
Central University of Tamilnadu
Thiruvarur.

Academic Coordinator

B. Tamilselvi,
Deputy Director, SCERT,
Chennai

Authors

M. Madhivanan
Headmaster Govt, Model Hr Sec
School , Karimangalam,
Dharmapuri

N. Kalaiselvam,
PGT, Chennai Girls HSS,
Nungambakkam, Chennai

A. Balamurugan,
PGT, Adhiyaman GBHSS,
Dharmapuri

C. S. Veeraragavan,
PGT, Sri Krishnaa Matric. Hr. Sec.
School, TVS Nagar, Coimbatore

S. Paneerselvam
PGT,(Rtd), GHSS, GKM Colony,
Chennai

R. M. Meenakshi
PGT Anna Adarsh MHSS Annanagar ,
Chennai

Dr. R. Swamynathan
Principal, Vidhya Giri Higher
Secondary School,
Puduvayal Karaikudi

Coordinator

K. P. Sujatha,
B.T Assistant,
GGHSS, K.V. Kuppam, Vellore-Dt

Content Readers

Dr. M. Alphonse,
Headmaster (Retd),
GHSS, Sadurangapattinam,
Kanchipuram Dt

G. Rajendiran,
Headmaster (Retd)
Thanthai Periyar GHSS,
Puzhithivakkam,
Kanchipuram Dt

ICT Coordinator

D. Vasu Raj
B.T. Asst (Retd),
Kosappur, Puzhal Block, Tiruvallur Dt

QR Code Management Team

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Pums, Ganesapuram- Polur

N. Jagan, Gbhss,
Uthiramerur, Kancheepuram

J. F. Paul Edwin Roy,
Pums, Rakkipatti, Salem

Art and Design Team

Illustration, Layout & Typist

S. Manoharan,
V.V. Graphics,
Palavanthangal,
Chennai - 114

In-House QC

Manohar Radhakrishnan
Gopu Rasuvel, Rajesh Thangappan
Jerald Wilson, Yogesh

Wrapper Design

Kathir Arumugam

Coordination

Ramesh Munisamy

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Printed by offset at:

14_11th-Maths-Vol2-Glossary.indd 284 09-04-2020 11:43:41